ALGEBRA II (spring 2023)

EXAM (Mo 15.5.2023 16-19 room CK112)

NOTE: choose 4 out of the 5 questions below!

All the fields below are subfields of the complex numbers.

1. Assume that G_1 and G_2 are groups. Define a binary operation (we use multiplicative notation) on the product set $G_1 \times G_2$ by setting for arbitrary $(g_1, g_2), (g_1', g_2') \in G_1 \times G_2$

$$(g_1, g_2)(g_1', g_2') := (g_1 g_1', g_2 g_2').$$

Prove that it makes $G_1 \times G_2$ a group.

(b) Consider the subgroup $H_1 := \{(g_1, 1_{G_2}) \mid g_1 \in G_1\} \leq G_1 \times G_2$. Is it a normal subgroup?

2. Let G be a group. We say that $f : G \to G$ is an inner automorphism if there is a fixed $g \in G$ such that $f(x) = g^{-1}xg$ for all $x \in G$. Show that the subgroup of inner automorphisms is a normal subgroup of the automorphism group of G.

3. True or false? Substantiate your answer e.g. by a short argument (you may use all the results proven during the course) or by a counterexample.

a) Let $\alpha \in \mathbf{C}$ and $\beta \in \mathbf{C}$. Then $|\mathbf{Q}(\alpha, \beta) : \mathbf{Q})| \leq |\mathbf{Q}(\alpha) : \mathbf{Q})| \cdot |\mathbf{Q}(\beta) : \mathbf{Q})|$

b) Every finite field extension is normal.

c) If a polynomial f has integer coefficients and is irreducible over $\mathbf{Z}$, then it is irreducible over $\mathbf{Q}$.

d) For every irreducible polynomial over $\mathbf{Q}$ the degree of the polynomial is a prime number.

4. a) Denote by L the normal closure of the extension $\mathbf{Q}(\sqrt[3]{2}) : \mathbf{Q}$. Determine L.

b) What is the Galois group of the extension $L : \mathbf{Q}$ (not all details are required)?

5. Prove the *third isomorphism theorem for groups*: Let G, H and K be groups such that $H \trianglelefteq G$, $K \trianglelefteq G$ and $K \leqslant H$. Then $H/K \trianglelefteq G/K$ and

$$(G/K)/(H/K) \cong G/H.$$

[Hint: Try to find as usual a suitable homomorphism between suitable objects!]