Matemaattisten tieteiden kandiohjelma / Mathstat
Bayesian inference
General exam 7.8.2019 (duration 3h)

No extra accessories allowed.

1. Assume i.i.d sample $Y_1, \dots, Y_n$ from the Pareto distribution Pareto(α, x_0), where the scale parameter x_0 is a known constant. A conjugate prior for the shape parameter α is a gamma distribution; now the full model is

$$\begin{aligned} Y_i \,|\, \alpha &\sim \text{Pareto}(\alpha, x_0) \quad \text{for all } i = 1, \dots, n \\ \alpha &\sim \text{Gamma}(a, b). \end{aligned}$$

Derive the posterior distribution $p(\alpha \,|\, \mathbf{y})$ for the parameter α given all the observations $\mathbf{y} = (y_1, \dots, y_n)$.

2. Consider one new observation $\tilde{Y} \,|\, \alpha \sim \text{Pareto}(\alpha, x_0)$ from the same distribution as the observations of the previous exercise. Assume also that this new observation is conditionally independent from the original observations given the parameter α. Denote the parameters of the posterior distribution of the previous exercise as a_n and b_n (assume the same gamma prior Gamma(a, b) from the previous exercise).

Show that the posterior predictive distribution for the new observation can be written as:

$$p(\tilde{y} \,|\, \mathbf{y}) = \frac{b_n^{a_n} a_n}{\tilde{y}\left(b_n + \log \frac{\tilde{y}}{x_0}\right)^{a_n + 1}}$$

3. Assume i.i.d. observations $Y_1, \dots, Y_n$ from the uniform distribution with both parameters a and b unknown. The conjugate prior for the parameter vector (a, b) is the bilateral bivariate Pareto distribution, so that the full model becomes:

$$\begin{aligned} Y_i \,|\, a, b &\sim \text{U}(a, b) \quad \text{for all } i = 1, \dots, n \\ a, b &\sim \text{Biv-Pareto}(\alpha, q, r). \end{aligned}$$

(a) Show that given all the observations the joint posterior $p(a, b \,|\, \mathbf{y})$ is Biv-Pareto(α_n, q_n, r_n) with the following parameters:

$$\begin{aligned} \alpha_n &= \alpha + n \\ q_n &= \min\{q, y_1, \dots, y_n\} \\ r_n &= \max\{r, y_1, \dots, y_n\}. \end{aligned}$$

Hint: You need to consider carefully the domains (sets in which they have non-zero values) of the density functions (you can for example use the indicator functions).

(b) Show that the marginal posteriors for the parameters a and b are:

$$\begin{aligned} p(a \,|\, \mathbf{y}) &= \frac{\alpha_n (r_n - q_n)^{\alpha_n}}{(r_n - a)^{\alpha_n + 1}} \quad \text{when } a < q_n \\ p(b \,|\, \mathbf{y}) &= \frac{\alpha_n (r_n - q_n)^{\alpha_n}}{(b - q_n)^{\alpha_n + 1}} \quad \text{when } b > r_n. \end{aligned}$$

4. Explain briefly what the following concepts mean, especially from the Bayesian inference viewpoint. Exact mathematical definitions are not required.

(a) Conjugate prior

(b) Informative prior

(c) Hyperparameter

(d) Bayes factor

(e) Empirical Bayes

Density functions

- The density function of the random variable X following the **Pareto distribution** Pareto(α, x_0) where the parameters $\alpha > 0, x_0 > 0$, is

$$p(x \mid \alpha, x_0) = \frac{\alpha x_0^\alpha}{x^{\alpha+1}}, \quad \text{when } x \geq x_0.$$

- The density function of the random variable X following the **gamma distribution** Gamma(α, β), where the parameters $\alpha > 0, \beta > 0$, is:

$$p(x \mid \alpha, \beta) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} \exp(-\beta x), \quad \text{when } x > 0.$$

- The density function of the random vector (A, B) following the **bilateral bivariate Pareto distribution** Biv-Pareto(α, q, r), where the parameters $\alpha > 1$ and $q < r$, is:

$$p(a, b \mid \alpha, q, r) = \frac{\alpha(\alpha+1)(r-q)^\alpha}{(b-a)^{\alpha+2}} \quad \text{when } a < q \text{ and } b > r.$$

- The density of the random variable X following the **uniform distribution** U(a, b) is:

$$p(x \mid a, b) = \frac{1}{b-a}, \quad \text{when } a < x < b,$$

where $a < b$.

Integrals

- Gaussian integral:

$$\int_{-\infty}^{\infty} e^{-x^2}\, dx = \sqrt{\pi}.$$

- Gamma function:

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t}\, dt.$$

- Beta function:

$$B(a, b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} = \int_0^1 t^{a-1}(1-t)^{b-1}\, dt.$$

- Properties of the gamma function:
  - $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$.
  - for all $x > 0$ it holds that: $\Gamma(x+1) = x\Gamma(x)$.