

Choose only 4 out of the following 6 questions!

1. Prove the Casorati-Weierstrass theorem: If f is a meromorphic function in the punctured ball $B(z_0, R) \setminus \{z_0\}$, and z_0 is an essential singularity for f , then for every $w \in \mathbb{C}$ and $\varepsilon > 0$ there is $z \in B(z_0, R) \setminus \{z_0\}$ such that $|f(z) - w| \leq \varepsilon$.
2. (i) Prove the converse direction in Montel's theorem, i.e. that a normal family $\mathcal{F}$ of analytic functions on the domain Ω is uniformly bounded on any compact subset $K \subset \Omega$.
 (ii) What is the largest domain $\Omega \subset \mathbb{C}$ on which the family $\mathcal{F} : \{z^n : n = 1, 2, \dots\}$ is normal?
3. (i) Define the order of an entire holomorphic function $f : \mathbb{C} \rightarrow \mathbb{C}$.
 (ii) Give examples of a (nontrivial) entire function having orders $0, 1, 2, \infty$. Substantiate your examples by a short explanation (not all details are needed).
 (iii) Use Hadamard's factorization theorem to show that for an entire function f of finite order the complement of the image $\mathbb{C} \setminus f(\mathbb{C})$ can contain at most one point.
4. Sketch the proof (give all the main ideas, not all details are required) of the following claim:
 If Ω is simply connected, and f is analytic in Ω with $f(z) \neq 0$ for all $z \in \Omega$, then one may define an analytic branch of $\log f(z)$ in Ω .
5. Let $u : \mathbb{C} \rightarrow \mathbb{R}$ be harmonic. Prove that for any $z \in \mathbb{C}$ and $r > 0$ one has the mean value property

$$u(z) = \frac{1}{2\pi} \int_0^{2\pi} u(z + re^{it}) dt.$$
 [Hint: you may assume this for analytic functions.]
6. Let $u : \Omega \rightarrow \mathbb{R}$ be harmonic in a convex domain $\Omega \subset \mathbb{C}$. Prove that the u has a harmonic conjugate v defined in the whole domain Ω .

COMPLEX ANALYSIS 2

EXAM

May 21 2025

- Write your name and student number to the upper-right corner of every sheet you return.
- Calculators, phones, books, computers or notes are not allowed. Please silence your phones.
- **Choose 4 out of the following 6 questions**, each of which is worth 6 points.

The maximum number of points for the whole exam is $4 \times 6 = 24$.

- Exam time is 3 hours. Good luck!