

COMPLEX ANALYSIS 2

EXAM

May 5 2025

- Write your name and student number to the upper-right corner of every sheet you return.
- Calculators, phones, books, computers or notes are not allowed. Please silence your phones.
- **Choose 4 out of the following 6 questions**, each of which is worth 6 points.
The maximum number of points for the whole exam is $4 \times 6 = 24$.
- Exam time is 3 hours. Good luck!

Choose only 4 out of the following 6 questions!

1. Prove that the series $\lim_{N \rightarrow \infty} \sum_{n=-N}^N (z-n)^{-1}$ defines a meromorphic function in the whole complex plane.
2. (i) Recall that the Gamma function was defined by the formula $\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt$ for $\operatorname{Re} z > 0$. Derive the basic functional equation $\Gamma(z+1) = z\Gamma(z)$.
(ii) Use the functional equation to extend Γ to a meromorphic function in the whole complex plane. Determine the poles and residues of the Gamma function.
3. What is a normal family of holomorphic functions? What is the largest domain $\Omega \subset \mathbf{C}^+ := \{z : \operatorname{Im} z > 0\}$ on which the family

$$\mathcal{F} : \{e^{nz^2} : n = 1, 2, \dots\}$$

is normal?

4. Sketch the main ideas of the proof (no details required) of the following claims:
 - (i) Annulus $\{1 < |z| < 2025\}$ is not conformally equivalent with the unit disc $\mathbf{D} := \{z \in \mathbf{C} : |z| < 1\}$.
 - (ii) If Ω is simply connected, and f is analytic in Ω with $f(z) \neq 0$ for all $z \in \Omega$, then one may define an analytic branch of $\log f(z)$ in Ω .
5. (i) Show that the function $z \mapsto \log |z|$ is harmonic in $\mathbf{C} \setminus \{0\}$.
(ii) Let u be a harmonic function in the annulus $A = \{z : r < |z| < R\}$ that only depends on the modulus $|z|$. Show that $u(z) = a \log(|z|) + b$, where a and b are constants.
6. Let $u : \Omega \rightarrow \mathbf{R}$ be harmonic in a convex domain $\Omega \subset \mathbf{C}$. Prove that the u has a harmonic conjugate v defined in the whole domain Ω .