Fourier-analysis II

Exam, 14.05.2018

1. Give the statement of the Riemann-Lebesgue lemma concerning the Fourier–transform of an $L^1(\mathbb{R}^d)$–function, and give also its proof.

2. Which of the following claims are true, and which are false? Explain your answers.

   (a) If $f \in L^1(\mathbb{R}^d)$, then $\hat{f}$ is continuous.

   (b) The function $g(x) = e^{-|x|}$, $x \in \mathbb{R}^d$, belongs to $\mathcal{S}(\mathbb{R}^d)$.

   (c) If $f \in L^p(\mathbb{R}^d)$, $1 \leq p \leq \infty$, then $f \in \mathcal{S}(\mathbb{R}^d)$.

   (d) If $f \in L^\infty(\mathbb{R}^d)$, then $f \in \mathcal{S}'(\mathbb{R}^d)$, and its distributional Fourier–transofrm $\hat{f} \in L^1(\mathbb{R}^d)$.

$\hat{\varsigma} \in L^1$

$\hat{\hat{\varsigma}} \in C^\infty$

3. (From homework) Suppose the Fourier–transform of $f \in L^1(\mathbb{R})$ satisfies

$$|\hat{f}(\xi)| \leq C(1 + |\xi|)^{-1-\alpha}, \ \xi \in \mathbb{R},$$

for some constants $0 < \alpha < 1$ and $C \geq 0$. Show that then f satisfies the Hölder–condition

$$|f(x) - f(y)| \leq M|x - y|^\alpha, \ x, y \in \mathbb{R}.$$

$\langle \hat{\hat{\varsigma}}, \varphi \rangle = \langle \varsigma, \hat{\hat{\varphi}} \rangle = c\langle \varsigma, \varphi^- \rangle = c\langle \varsigma^-, \varphi \rangle$

4. Assume that $0 \neq f \in L^1(\mathbb{R}^d)$ and that $\hat{f} = \lambda f$ for some complex constant λ. What can you say of λ? Can you give an example of a non–zero function f satisfying this relation?