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FUNCTIONAL ANALYSIS II EXAM 1 November 8th, 2022

Solutions are accepted in English, Finnish or Swedish.

1. Note! You are expected to answer to only one of the cases a) or b); choose as you wish.

a) Let $\Omega = \mathbb{R}$. Write the Dirac measure $\delta_{-2} \in \mathcal{D}'(\Omega)$ of the point $-2 \in \Omega$ as a distributional derivative of some continuous function $f : \mathbb{R} \to \mathbb{C}$.

b) Calculate the Fourier transform of the polynomial $P(x) = 1 + x^2$, where $x \in \mathbb{R}$ variable. Here, P is considered as a tempered distribution, i.e. an element of the space $\mathcal{S}'(\mathbb{R})$.

2. Show that the subspace $X = \{f \in C^\infty(\mathbb{R}) \mid f(5) = 0\}$ of the space $C^\infty(\mathbb{R})$ is closed, in other words, if $(f_n)_{n=1}^\infty$ is a sequence of elements of X converging to a function $g \in C^\infty(\mathbb{R})$, then $g \in X$.
The topology of $C^\infty(\mathbb{R})$ is defined by the seminorms

$$p_m(h) = \sup_{\substack{-m \le x \le m \\ k \le m}} \left| \frac{d^k h(x)}{dx^k} \right|,$$

where $m = 1, 2, \ldots$

3. Let $T \in \mathcal{D}'(\mathbb{R})$. How do you define the distribution fT, where $f(x) = \sin x$, $x \in \mathbb{R}$? Is the following formula valid,

$$\frac{d(fT)}{dx} = \cos x \, T + \sin x \frac{dT}{dx},$$

where the derivatives are distributional.