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**FUNCTIONAL ANALYSIS II EXAM 2 December 20th, 2022**

Solutions are accepted in English, Finnish or Swedish. All problems 1,2 and 3 are worth 8 points max.

**1. In the following items 1.1-1.12, just answer "yes", if the claim is true, or "no" in the opposite case. No explanations or proofs are needed.**

(*Recall that given a bounded, closed interval* $[a, b] \subset \mathbb{R}$, *the space* $C^1([a, b])$ *consists of those continuously differentiable functions* $f : (a, b) \to \mathbb{C}$, *for which* $f$ *and its derivative can be extended continuously and boundedly to* $[a, b]$. *The space* $C_B^1(a, b)$ *consist of those continuously differentiable functions* $f : (a, b) \to \mathbb{C}$, *for which* $f$ *and its derivative are bounded on* $(a, b)$.)

1.1) There exists $\lambda$ with $0 < \lambda < 1$ such that the Hölder space $C^{0,\lambda}([-1, 1])$ is contained in $C^1([-1, 1])$.

1.2) The space $C^1([-1, 1])$ is not contained in the space $C_B^1(-1, 1)$.

1.3) The identity operator $I : W^{2,2}(\Omega) \to L^\infty(\Omega)$ is well defined and continuous, when $\Omega = (0, 1) \subset \mathbb{R}$ .

1.4) The identity operator $I : W^{1,0}(\mathbb{R}^2) \to L^\infty(\mathbb{R}^2)$ is well defined and continuous.

1.5) For a domain in $\Omega \subset \mathbb{R}^n$, $2 \le n \in \mathbb{N}$, the cone property implies the strong local Lipschitz property

1.6) The domain $\Omega = \{(x, y) \in \mathbb{R}^2 : x > 0, y > 0\} \subset \mathbb{R}^2$ has the cone property.

1.7) The domain $\Omega = \{(x, y) \in \mathbb{R}^2 : 0 < x < y^2, y > 0\} \subset \mathbb{R}^2$ has the cone property.

1.8) The partial differential operator $\frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2}$ is elliptic in the domain $\mathbb{R}^2 \ni (x_1, x_2)$

1.9) The partial differential operator $-\frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2}$ is elliptic in $\mathbb{R}^2 \ni (x_1, x_2)$.

1.10) The partial differential operator $-\left(\frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_2}\right)^2$ is elliptic in $\mathbb{R}^2 \ni (x_1, x_2)$.

1.11) If $\Pi$ is a planar domain, which is 1-periodic in the $x_1$-direction, and $\varpi$ is its periodic cell, the Floquet transform $\mathcal{F}$ maps the space $L^2(\Pi)$ into a Bochner-type space (vector valued $L^2$-space) defined on $I = (-\pi, \pi)$, denoted by $L^2(I; L^2(\varpi))$.

1.12) The mapping $\mathcal{F}$ is a surjection in the spaces mentioned in 1.11.

**2.** Let $p = 2$ and $n = 3$, and let $q \in \mathbb{R}$ satisfy $2 < q < 6$. Then, it is known by the embedding theorem that the identity operator $I : W^{1,p}(\mathbb{R}^n) \to L^q(\mathbb{R}^n)$ is well defined and continuous. Prove that $I$ is not compact (which means that the unit ball of the Banach space $W^{1,p}(\mathbb{R}^n)$ is not a precompact subset of $L^q(\mathbb{R}^n)$, or, that there exists a sequence $(f_\ell)_{\ell \in \mathbb{N}} \subset W^{1,p}(\mathbb{R}^n)$ such that $\|f_\ell\|_{p,1} = 1$ for all $\ell$ but no subsequence converges in $L^q(\mathbb{R}^n)$ ).

**3.** Let $\Omega := (0, 5) \subset \mathbb{R}$ and $1 < p < \infty$. Construct a bounded linear extension operator $E : W^{1,p}(\Omega) \to W^{1,p}(\mathbb{R})$ (the operator $E$ must be such that $(Ef)(x) = f(x)$ for all $f \in W^{1,p}(\Omega)$, for almost every $x \in \Omega$ , i.e. $E$ extends the functions $f \in W^{1,p}(\Omega)$ to the whole set $\mathbb{R}$.)