MAST31002 Functional analysis Second partial exam (2,5 h) 11.1.2023

In addition to standard writing equipment, you are allowed to bring in and consult **one handwritten two-sided** *A4 sheet of personal notes during the exam.* Please include your *name* and *student number* on your answer sheets.

Problem 1

Suppose E is a normed space over the scalar field $\mathbb{K}$. Assume that E is *not* complete.

(a) Denote $X := \mathcal{L}(E)$, and take $A, B \in X$ and $\alpha \in \mathbb{K}$. Write down the (standard) definitions of $A + B \in X$, $\alpha A \in X$ and $\|A\|$. Is X always a Banach space? (You do not need to prove your answer.)

(b) Consider the (continuous) dual E^* of E. Give the definition of E^*, including its norm. Is E^* always a Banach space? (You do not need to prove your answer.)

(Grading: *3 points* from each of the items.)

Problem 2

Suppose H is a Hilbert space, with an inner product $(\cdot|\cdot)$. Assume that M is a closed linear subspace of H.

(a) Give the definition of the orthogonal complement $M^\perp$ of M. Give the definition of the orthogonal projection P_M onto M. (*2 points*)

(b) Assume that F is a normed space, and $T_1 \in \mathcal{L}(M, F)$. Show that T_1 has a continuous linear extension to H, i.e., show that there is $T : H \to F$ which is continuous, linear, and $Tx = T_1 x$ for all $x \in M$. Can you find such an extension if M is *not* closed? (*3 points*)

(c) Give an example which proves that the extension T in (b) is not always unique. (*1 point*)

Problem 3

Consider $E := C([0,1])$ endowed with the sup-norm which was proven to be a Banach space during the course. Given $f \in E$, define $Sf : [0,1] \to \mathbb{R}$ by setting

$$(Sf)(x) = f(x) - \int_0^x t\, f(t)\, \mathrm{d}t\,, \qquad x \in [0,1]\,.$$

(a) Show that $S : f \mapsto Sf$ is a continuous linear map $E \to E$. Prove using the Neumann series that S is an invertible operator, i.e., that it has an inverse map S^{-1} and the inverse map is a bounded operator. (*4 points*)

(b) Show that the inverse operator S^{-1} is positivity preserving: If $f(x) \geq 0$ for all x, then also $(S^{-1}f)(x) \geq 0$ for all x. Is the original operator S also positivity preserving? (*2 points*)

Problem 4

(a) Write down the assumptions and statement of the Open Mapping Theorem. (*2 points*)

(b) Assume that a_k, $k \in \mathbb{N}$, are real numbers with the following property: *the series* $\sum_{k=1}^{\infty} a_k x_k$ *is convergent for every real sequence* $x \in \ell^1$. Given $x \in \ell^1$, let $T(x)$ denote the value of the series and consider the function $T : \ell^1 \to \mathbb{R}$. Prove that $\sup_k |a_k| < \infty$, $T \in (\ell^1)^*$, and that $\|T\| = \sup_k |a_k|$. (*4 points*)

(*Hint:* Banach Steinhaus theorem.)