MAST31002 Functional analysis First partial exam (2,5 h) 24.10.2022

Please include your *name* and *student number* on your answer sheets. In addition to standard writing equipment, you are allowed to bring in and consult **one** handwritten two-sided A4 sheet of personal notes during the exam.

General advice: Even for a maximal grade, you do not need to be able to answer all of the items below correctly. If an item looks tricky, it is better to jump over it and, time allowing, return to it later.

Problem 1

The following five standard sequence and function spaces have been introduced during the course

$$\mathbb{R}^4, \quad \ell^{\frac{3}{4}}, \quad \ell^4, \quad L^2(\mathbb{R}^4), \quad L^\infty(\mathbb{R}^2).$$

(a) From the above list, select all Banach spaces and give in detail the definition of the *space* and of its *norm*. (You do not need to justify your answer.)

(b) From the above list, select all Hilbert spaces and give in detail the definition of the *space* and of its *inner product*. (You do not need to justify your answer.)

(Grading: in total *6 points* from the two items.)

Problem 2

(a) Consider the vector spaces ℓ^p, ℓ^q consisting of real-valued sequences, for some p, q satisfying $1 < p, q < \infty$. Suppose $f \in \ell^p$ and $g \in \ell^q$. Write down the assumptions and the statement of the Hölder's inequality relevant to f and g. *(2 points)*

(b) Let $1 \leq p < p' \leq \infty$. Show that for all $x = (x_n) \in \ell^p$ we have also $x \in \ell^{p'}$ and

$$\|x\|_{p'} \leq \|x\|_p \,. \qquad \textit{(4 points)}$$

Problem 3

Consider the complex function space

$$C_0(\mathbb{R}_+) := \left\{ f : \mathbb{R}_+ \to \mathbb{C} \,\middle|\, f \text{ continuous and } \lim_{x\to\infty} f(x) = 0 \right\},$$

where $\mathbb{R}_+ := [0, \infty[= \{x \in \mathbb{R} \,|\, x \geq 0\}$.

(a) Show that $C_0(\mathbb{R}_+) \subset BC(\mathbb{R}_+) := \{f : \mathbb{R}_+ \to \mathbb{C} \,|\, f \text{ continuous and bounded}\}$. *(3 points)*

(b) It was proven during the lectures that $BC(\mathbb{R}_+)$ is a Banach space under the sup-norm, $\|f\|_\infty := \sup_{x\geq 0} |f(x)|$. Show that also $C_0(\mathbb{R}_+)$ is a Banach space under the sup-norm. *(3 points)*

Problem 4

(a) Write down the assumptions and statement of the Banach fixed point theorem. *(2 points)*

(b) Show that there is exactly one function $f_0 \in C_0(\mathbb{R}_+)$ for which $|f_0(x)| \leq 1$ for all x and

$$f_0(x) = \frac{1}{\sqrt{4+x}} + \frac{1}{4}\int_0^\infty e^{-y} f_0(yx) f_0(yx^2) \mathrm{d}y\,, \quad x \geq 0\,. \qquad \textit{(4 points)}$$