Department of Mathematics and Statistics
Homotopy theory
Course exam 8.2.2024

1. a) Calculate $\pi(X)$, where $X = \mathbb{R}^3 \setminus \{(0,0,z) \in \mathbb{R}^3 \mid z \in \mathbb{R}\}$.
b) Calculate $\pi(X)$, where $X = \mathbb{R}^3 \setminus \{(0,0,z) \in \mathbb{R}^3 \mid z \geq 0\}$.

2. a) Suppose that $f: S^1 \to S^1$ is a continuous map and $f \simeq \mathrm{id}_{S^1}$. Prove that f is surjective.
b) Let $\gamma_0, \gamma_1: I \to S^2$ be paths, where

$$\gamma_0(t) = (\cos 2\pi t, \sin 2\pi t, 0) \quad \text{and} \quad \gamma_1(t) = (0,0,1),$$

for every $t \in I$. Explicitly construct a homotopy

$$F: I \times I \to S^2$$

from γ_0 to γ_1.

3. a) Explain how knowledge about the fundamental group of a space Y helps us to classify all possible covering spaces over Y. Introduce the results which are needed (no proofs required).
b) Use the circle S^1 as an example: Give a list of all possible covering spaces over S^1 (up to isomorphism). How do we know that the list is complete and doesn't contain any covering spaces, which are isomorphic with each other?

4. a) What are the homotopy groups $\pi_k(S^n)$ for $n \geq 1$, $1 \leq k \leq n$ (no proofs required). (2 p.)
b) Using the information above, prove in detail that the spaces $\mathbb{R}^m$ and $\mathbb{R}^n$ are not homeomorphic, when $m, n \geq 2$, $m \neq n$. (4 p.)

5. Give the definition of an exact sequence of groups. Explain briefly the construction of the exact homotopy sequence of a weak fibration (no proofs required). Give an example, how looking at the exact sequence, we can get new information concerning homotopy groups of certain spaces.