Department of Mathematics and Statistics
Homotopy theory
2nd midterm exam 18.12.2023

1. a) What are the homotopy groups $\pi_k(S^n)$ for $n \geq 1$, $1 \leq k \leq n$ (no proofs required). (2 p.)

b) Using the information above, prove in detail that the spaces $\mathbb{R}^m$ and $\mathbb{R}^n$ are not homeomorphic, when $m, n \geq 2$, $m \neq n$. (4 p.)

2. a) Let X be a topological space and $A \subset X$ a closed subspace. Prove that the inclusion $A \to X$ is a cofibration, if and only if $(X \times \{0\}) \cup (A \times I)$ is a retract of $X \times I$.

b) Prove that the inclusion $\{\frac{1}{2}\} \to [0, 1]$ is a cofibration.

3. a) Prove that the composite of two fibrations is a fibration.

b) Suppose that $p: E \to B$ is a fibration and $f_1, f_2: X \to B$ are continuous maps. Suppose also that $f_1 \simeq f_2$. Prove that the map f_1 has a lift $\tilde{f}_1: X \to E$ if and only if the map f_2 has a lift $\tilde{f}_2: X \to E$.

4. Give the definition of an exact sequence of groups. Explain briefly the construction of the exact homotopy sequence of a weak fibration (no proofs required). Give an example, how looking at the exact sequence, we can get new information concerning homotopy groups of certain spaces.