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Homotopy theory, MAST 31020
Midterm Exam, March 5, 2019

1. (6 points) Consider the diagram

$$\begin{array}{ccc} A & \xrightarrow{h} & B \\ {\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle g} \\ C & \xrightarrow[k]{} & D \end{array}$$

in the category **Top**. Suppose the diagram is a pushout and f is a homeomorphism. Show that also g is a homeomorphism.

2. (6 points)

(1) Find the fundamental groups of the following spaces:
  (a) $\mathbb{R}P^3 \times \mathbb{S}^1$
  (b) $\mathbb{R}P^2 \vee \mathbb{S}^1$

(2) Let $T = \mathbb{S}^1 \times \mathbb{S}^1$ be the torus. Find three coverings $p_i \colon E_i \to T$, $i = 1, 2, 3$, such that E_i and E_j are not homeomorphic for $i \neq j$.

3. (6 points) Let $\tilde{X}$ and $\tilde{Y}$ be simply-connected covering spaces of the path-connected, locally path-connected spaces X and Y, respectively. Assume X and Y are homotopy equivalent. Show that $\tilde{X}$ and $\tilde{Y}$ are homotopy equivalent.

4. (6 points)

(1) Let X and Y be topological spaces, and let $\mathrm{C}(X, Y)$ denote the space of continuous maps $X \to Y$ equipped with the compact open topology. Let K be a compact subset of X, and let V and U be open subsets of Y such that $\overline{V} \subset U$. Show that

$$\overline{(K; V)} \subset (K; U),$$

where $\overline{(K; V)}$ denotes the closure of $(K; V)$ in $\mathrm{C}(X, Y)$.

(2) Let $I = [0, 1]$. Assume $\mathrm{C}(I, X)$ equipped with the compact open topology is path-connected. Show that X is path-connected.