Introduction to Algebraic Topology
Exam February 25, 2025
Exam time 12.14-14.00

Problems

In the following problems, $B^n(x,r) = \{y \in \mathbb{R}^n : |y-x| < r\}$ is an open Euclidean ball in $\mathbb{R}^n$ and $S^{n-1}(x,r) = \partial B^n(x,r) = \{y \in \mathbb{R}^n : |y-x| = r\}$ a Euclidean sphere. We also denote $B^n = B^n(0,1)$ and $S^{n-1} = S^{n-1}(0,1)$.

p1. Let X be a path-connected space, $x_0, x_1 \in X$, and let $f\colon X \to Y$ be continuous map for which the induced map $f_*\colon \pi_1(X,x_0) \to \pi_1(Y, f(x_0))$ is injective. Show that the induced map $f_*\colon \pi_1(X,x_1) \to \pi_1(Y, f(x_1))$ is injective. (*Hint:* Writing is easier if the induced maps have different names, like f_*^0 and f_*^1.)

p2. Let $X = S^1 \times S^1$. Give an example of a covering map $\varphi\colon \widehat{X} \to X$, where $\widehat{X}$ is path-connected and not homeomorphic to X, for which all lifts of the loop $\alpha\colon [0,1] \to X$, $t \mapsto (e^{4i\pi t}, (1,0))$, under φ are loops.

p3. Let $y_0 \in S^2$, let $Y = S^1 \times S^2$, let $X = Y \cup (B^2 \times \{y_0\}) \subset \mathbb{R}^5$, and $x_0 = ((1/2, 0), y_0) \in X$. Show that $\pi_1(X, x_0)$ is a trivial group. (*Hint:* Seifert–van Kampen with $\mathscr{U} = \{U, V\}$, where $U = B^2(0, 3/4) \times \{y_0\}$. Draw a picture.)