

INTRODUCTION TO ALGEBRAIC TOPOLOGY: EXAM 2

Write your answers as rigorously as possible, making sure to justify all of your steps. You can use any statements that were proved in the lectures, the homework, or in sections of the textbook that we covered. Be sure to cite the result you are using and explain why it applies. When asked to compute the homology of a space, please display your final answer in one place.

We use the standard notation S^n to refer to the n -sphere, $S^n = \{x \in \mathbb{R}^{n+1} \mid |x| = 1\}$.

Problem 1 (20 POINTS)

Let P be a space consisting of a single point, and let $C_\bullet(P)$ be its singular chain complex. Let $C_\bullet^{\leq 2m}(P)$ be the subcomplex of singular n -chains for $n \leq 2m$. That is,

$$C_n^{\leq 2m}(P) = \begin{cases} C_n(P) & \text{for } n \leq 2m \\ 0 & \text{else} \end{cases}$$

For $m \geq 0$, show that the inclusion $\iota : C_\bullet^{\leq 2m}(P) \hookrightarrow C_\bullet(P)$ is a chain homotopy equivalence.

Problem 2 (10 POINTS)

Let A be a closed subspace of a space X such that there exists an open set $U \subset X$ with $A \subset U$ and U deformation retracting onto A . Assume that $H_k(X) \cong H_k(S^n)$ and $H_k(A) \cong H_k(S^{n-1})$ for some $n \geq 1$ and for all k . Calculate the homology groups of the quotient space X/A .

Problem 3 (20 POINTS)

Take integers $n, m \geq 2$ with $m \neq n$, and $r \geq 1$. For $i = 1, \dots, r$, take distinct points $x_i \in S^n$ and distinct points $y_i \in S^m$. Let

$$X = S^n \sqcup S^m / \sim \quad \text{where } x_i \sim y_i \text{ for all } i = 1, \dots, r$$

be the space obtained by identifying each of the r points in S^n with the corresponding points in S^m . Calculate the homology groups of X .