

Introduction to Mathematical Biology

EXAM: 22 October 2024

1. The exam lasts 165 minutes (9.15-12.00).
2. On the first page, please write your name and email readably (e.g. in print).
3. Put your name on every separate paper.
4. You can ask me if you do not understand a question only because of the English.
5. I shall email you the results, and if you wish, you can have a look at the corrected exam. If you do not hear about the results within two weeks, please contact me.
6. You can re-take the exam if you want a better grade.

The problems are given equal weight in the grade.

1. Healthy individuals move to a pre-cancer state at a rate γ_1 , from where the condition progresses to cancer at a rate γ_2 . Everyone dies at a rate μ except those who have cancer; the latter, in addition to death at a rate μ due to other causes, die at an extra rate α due to cancer (their death rate is thus $\alpha + \mu$). All rates are constants and everyone is born healthy.

- (a) What is the probability that a newborn will die due to cancer?
- (b) If individuals with cancer do not reproduce and all others give birth to offspring at a constant rate b , what is the expected number of offspring per lifetime?

2. Suppose that population growth is limited by the available living sites and the offspring actively search for these sites for a short time period T after their birth. If an offspring moves at random, then the number of empty sites it could encounter during T time is Poisson distributed with expectation $\gamma T(M - N)$, where M is the number of all sites, $M - N$ is the number of sites not occupied by an individual, and γ is a constant of proportionality. The offspring settles into the first empty site it finds. The probability that it finds at least one empty site and therefore it survives is $1 - e^{-\gamma T(M-N)}$. Assuming that T is so short on the time scale of population growth that settling into an empty site or dying otherwise can be considered instantaneous, we arrive at the model

$$\frac{dN}{dt} = [b(1 - e^{-\gamma T(M-N)}) - \mu] N,$$

where b and μ are the constant birth and death rates, respectively. Find all equilibria and establish their stability.

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3. Suppose that infected individuals release virus particles into the environment at a *per capita* rate α and these virus particles decay at a rate δ . With I infecteds present, the number of virus particles P therefore follows the dynamics

$$\frac{dP}{dt} = \alpha I - \delta P.$$

The virus particles infect susceptible individuals according to mass action with rate β . Assuming that the total number N of individuals is constant and everyone not infected is susceptible (there is no immunity to the disease), the number of infecteds changes as

$$\frac{dI}{dt} = \beta(N - I)P - \nu I,$$

where ν is the rate of recovery. The number of virus particles P changes much faster than the number of infecteds I .

- (a) Calculate the quasi-equilibrium $\hat{P}$ and show that it is stable.
- (b) Show that the number of infecteds follows logistic growth ($\frac{dI}{dt} = r_0(1 - \frac{I}{K})I$) and determine its nontrivial equilibrium K .

4. Insect pests could be eradicated by releasing sterilized males: Females who mate with a sterile male do not produce any offspring, so that the average fecundity of females decreases. Suppose that there are N_t females and N_t males (equal sex ratio) and we release S sterile males in addition. Assuming that without control the population follows the Beverton-Holt model, the dynamics with control are given by

$$N_{t+1} = \frac{N_t}{N_t + S} \frac{\lambda N_t}{1 + aN_t},$$

where the first factor is the probability that a female mates with a fertile male. We assume $\lambda > 1$ so that without control, the population is viable.

- (a) Show that for any $S > 0$, the trivial equilibrium is stable.
- (b) This model can have at most two nontrivial equilibria. With increasing S , these two equilibria collide and disappear in a fold bifurcation, so that for $S > S_0$, the population is guaranteed to go extinct. Determine the S_0 where the fold bifurcation happens (the non-degeneracy conditions need not be checked).