INTRODUCTION TO NUMBER THEORY

Final Exam (Monday 9.12.19)

OBS. Answer only 4 of the following 6 questions (you are free to choose the 4 questions you answer). You may answer either in Finnish or English.

1. Find all integers that leave the remainder 1 when divided 3, the remainder 2 when divided by 5, and the remainder 3 when divided by 7.

2. (i) Define Euler's function φ. What is the formula for $\varphi(n)$ if one knows the prime number decomposition of $n \geq 2$, say $n = p_1^{\alpha_1} \ldots p_k^{\alpha_k}$, where the p_j:s are distinct primes.

(ii) Prove: if m and n are positive integers such that $m|n$, then also $\varphi(m)|\varphi(n)$.

3. (i) Let (x_0, y_0) be the smallest positive solution to Pell's equation $x^2 - Dy^2 = 1$. What is the second smallest positive solution in terms of x_0 and y_0?

(ii) Find the continued fraction expansion of $\sqrt{7}$. Determine the fundamental solution of Pell's equation $x^2 - 7y^2 = 1$.

4. Show (e.g. using theorems of the lectures or directly), that there is a constant $c > 0$ such that

$$|\sqrt{5} - \frac{m}{n}| \geq \frac{c}{n^2} \qquad \text{for all integers } m, n \quad (n \neq 0).$$

5. (i) Prove by assuming only the notions of divisibility and a prime number, that every positive integer $n \geq 2$ can be written as a product of prime numbers.

(ii) Describe (perhaps with not all details) how one proves the same for Gaussian integers, i.e., that every Gaussian integer (that is not 0, nor a unit) is a product of Gaussian primes.

6. (i) Define the following notions: Gaussian integers, Gaussian units, greatest common divisor of given Gaussian integers, Gaussian primes, associates of a given Gaussian integer. Give some examples.

(ii) Find all the Gaussian prime factors (up to moving to an associate) of 51.