MAST31401 Inverse Problems 1: convolution and deconvolution, 5 Cr.
Home exam, 13.10.2020. Return deadline: 23.10.2019 at 10:00.

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Please read the following instructions carefully before starting the exam:

- If you did not pass the exercises, then you cannot take to the exam. At least 50% of the points are needed to pass them.
- This home exam contains 3 problems. Each problem is worth 10 points. 15 points is required to pass the exam. Your final grade will depend on the total points you got from the exercises and the points you get from this exam.
- Grading is based on you showing you understand what you are doing, so explain the choices you make.
- Use for example LaTeX, Word etc to write your solutions. You can answer either in Finnish or English.
- Technical instructions. Create a single PDF file containing your written answers to all of the questions. The PDF should have all the needed figures in it. Remember to also send in the codes, in runnable .m-files, for the questions where we work with Matlab. Remember that this is also partly an exercise on scientific writing, so you should refer to the figures and formulas you have in your text. Please zip the other files, so send the PDF and a zip file containing the needed m-files.
- Return the PDF file and the m-files before the deadline 23rd October at 10:00am. You can ask for an extended deadline before 21.10.2020 if you have an acceptable reason. If you return your exam late, every 1 hour will reduce your result by 1 point. (E.g. return at 23.10.2020 at 10:01 will reduce your result by 1 point, at 11:01 by 2 points, etc.)

Exercise 1. Finnish Meteorological Institute (FMI) offers open data on its website. Shown in the top of Figure 1 are daily maximum temperatures $f \in \mathbb{R}^{365}$ measured in Kumpula during year 2019. The bottom image in Figure 1 shows the moving average of 9 consecutive data points; let's call this vector $m \in \mathbb{R}^{365}$. The files named in this exercise are in the ZIP-folder Exercise 1.

The goal of this part of the exam is to recover the original data f from the moving average m using several different methods.

1. The point spread function in this case is $p = [\frac{1}{9}, \frac{1}{9}, \frac{1}{9}, \frac{1}{9}, \frac{1}{9}, \frac{1}{9}, \frac{1}{9}, \frac{1}{9}, \frac{1}{9}]$. The MATLAB routine `Step02_SVD_comp.m` forms the measurement matrix A and calculates its singular value decomposition.

   Implement the truncated SVD reconstruction method. Which truncation index gives the smallest reconstruction error? (Measure the error as relative square norm error.)

2. The MATLAB routine `Step03_Tikhonov_comp.m` calculates the classical Tikhonov regularization

   $$\arg\min_x \{\|Af - m\|_2^2 + \alpha\|f\|_2^2\}.$$

   What is the minimal relative square norm reconstruction error you can get by varying $\alpha > 0$?

   Implement the variant of Tikhonov regularization formulated like this:

   $$\arg\min_x \{\|Af - m\|_2^2 + \alpha\|f - m\|_2^2\}.$$

   What is the idea in using the regularization term $\alpha\|f - m\|_2^2$ instead of $\alpha\|f\|_2^2$? What is the minimal relative square norm reconstruction error you can get by varying $\alpha > 0$?

3. The MATLAB routine `Step04_L1reg_comp.m` calculates the L^1-regularization

   $$\arg\min_x \{\|Af - m\|_2^2 + \alpha\|f\|_1\}.$$

   What is the minimal relative square norm reconstruction error you can get by varying $\alpha > 0$?

   Implement the variant of L^1-regularization formulated like this:

   $$\arg\min_x \{\|Af - m\|_2^2 + \alpha\|f - m\|_1\}.$$

   What is the minimal relative square norm reconstruction error you can get by varying $\alpha > 0$?

Exercise 2. This question requires you to write a small essay (about 2 pages). The topic of the essay is convolution and deconvolution, well-posedness and point spread functions. Examples of things to discuss are what these concepts are, formulations of the concepts, examples, what kind of effects they have, what kind of troubles you can have, how these concepts are related to each other and how they relate to linear inverse problems. You should discuss at least 10 points in total.

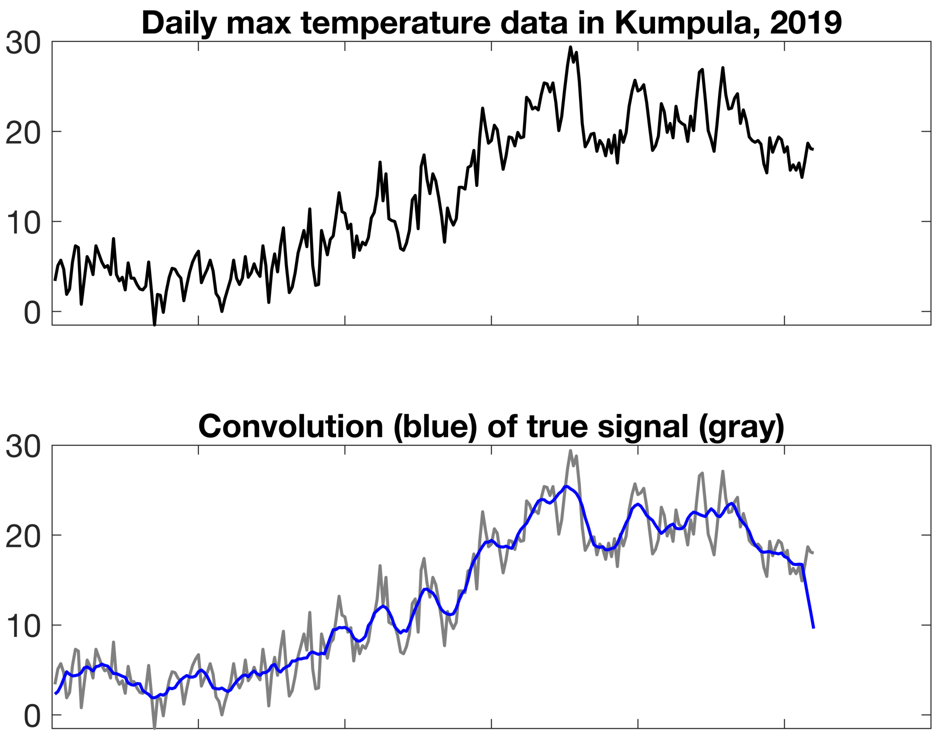

Figure 1: Top: daily maximum temperature data in Kumpula for year 2019. Bottom: 9-bin moving average shown in blue on top of the original data.

Exercise 3. This exercise relies on the material contained in the compressed folder `Exercise3.zip` attached to the question set. In order to use it, download the compressed folder and extract the file contained in it, `Exercise3.mat`. Make sure that such file is copied in the current directory in which you are running Matlab.

The file `Exercise3.mat` contains four variables, which can be uploaded to the workspace via the command `load('Exercise3.mat')`:

- `md` is a blurred and noisy signal, associated to an unknown original signal `f`.
- `PSF` is the point spread function responsible for the blurring of the signal.
- `delta` is the level of the noise acting on `md`.
- `prior` is a variable containing an important prior information on `f`: namely, `prior` is a vector which we assume to be similar to the unknown signal `f`.

To summarize, you have received some blurred and noisy measurements `md` such that $\|\mathtt{md} - \mathtt{m}\| \leq \mathtt{delta}$, where `m` is the convolution between an unknown signal `f` and the known filter `PSF`.

Use the provided data to recover the original signal `f`, employing the regularization theory of inverse problems. In particular,

1. We are working with three different regularization strategies: Tikhonov regularization, Generalized Tikhonov regularization promoting smoothness (derivative) and Generalized Tikhonov with the knowledge of a signal close by (`prior`). Write the formulation of the three strategies and provide a full description of all the variables appearing in the formulations.
2. Select the best value α to be employed in Tikhonov. Among the possible methods for the optimal choice of α, use the one involving the knowledge of `delta`.
3. Select the best value α to be employed in Generalized Tikhonov promoting small norms of the derivative. Among the possible methods for the optimal choice of α, use the one involving the knowledge of `delta`.
4. Select the best value α to be employed in Generalized Tikhonov promoting closeness to `prior`. Among the possible methods for the optimal choice of α, use the one involving the knowledge of `delta`.
5. In each of the three cases, report the optimal value of α and show the deconvolved signal obtained with that choice. Describe the main differences between the three deconvolved signals. Remember that you have prior information of the signal.