University of Helsinki
Department of Mathematics and Statistics
Introduction to Logic 1
Course Examination, Solutions
March 3, 2020

1. Use resolution to show that $p_0 \leftrightarrow p_2$ is a logical consequence of the assumptions $p_0 \leftrightarrow p_1$ and $p_1 \leftrightarrow p_2$

   Solution:

   First we transform the assumptions to conjunctive normal form, for example by using a truth table.

p_0	p_1	$(p_0 \leftrightarrow p_1)$
1	1	**1**
1	0	**0**
0	1	**0**
0	0	**1**

   So $p_0 \leftrightarrow p_1$ is equivalent to $(\neg p_0 \vee p_1) \wedge (p_0 \vee \neg p_1)$.
   From this we get the clauses $\{\neg p_0, p_1\}$ and $\{p_0, \neg p_1\}$.
   Similarly, from $p_1 \leftrightarrow p_2$ we get the clauses $\{\neg p_1, p_2\}$ and $\{p_1, \neg p_2\}$.

   The negation of the conclusion $\neg(p_0 \leftrightarrow p_2)$ written as clauses becomes $\{p_0, p_2\}$ and $\{\neg p_0, \neg p_2\}$.

   Now we can do the resolution:

   1. $\{\neg p_0, p_1\}$
   2. $\{\neg p_1, p_2\}$
   3. $\{\neg p_0, p_2\}$ from 1. and 2.
   4. $\{p_0, p_2\}$
   5. $\{p_2\}$ from 3. and 4.
   6. $\{p_1, \neg p_2\}$
   7. $\{p_0, \neg p_1\}$
   8. $\{p_0, \neg p_2\}$ from 6. and 7.
   9. $\{\neg p_0, \neg p_2\}$

10. $\{\neg p_2\}$ from 8. and 9.

11. $\emptyset$ from 5. and 10.

It is also possible to derive the clauses $\{\neg p_0, p_2\}$ and $\{p_0, \neg p_2\}$ and argue that they represent the formula $p_0 \leftrightarrow p_2$.

Grading: Max 6 points. Reduce up to 2 points if clauses are not formed correctly. Reduce 2 points each time the resolution rule is not used correctly. Reduce 1 or 2 points if there are missing steps or many steps are combined into one. Reduce 3 points if the resolution is not carried out completely and therefore the desired conclusion is not reached.

If only the clauses are formed correctly, give 3 points. If the resolution proof goes completely wrong, give 1 or 2 points if there are some correct applications of rules or right beginning for the solution.

2. Use natural deduction to derive $\neg B$ from A and $\neg A \lor \neg B$.

   Solution:

   To derive $\neg B$, we finally use disjunction elimination rule. $\neg B$ is conveniently the other disjunct and for the case of $\neg A$ we use the given assumption A and temporary assumption $\neg A$ to derive a contradiction:

$$\dfrac{\neg A \lor \neg B \qquad \dfrac{\dfrac{A \qquad [\neg A]^1}{A \land \neg A}\,{\scriptstyle \land I}}{\neg B}\,{\scriptstyle \neg I} \qquad [\neg B]^1}{\neg B}\,{\scriptstyle \lor E, 1}$$

   There are other possible natural deductions. This may be the shortest one.

   Grading: Max 6 points. Reduce 2 points each time a rule is used incorrectly. Reduce 1 or 2 points from unclear presentation. Reduce 2 points if assumptions that need to be closed are left unclosed.

   If the natural deduction goes completely wrong, give 1 or 2 points if there are some correct applications of rules or right beginning for the solution.

3. Give a semantic proof of

$$(A \vee (B \wedge C)) \rightarrow ((A \vee B) \wedge (A \vee C)).$$

Solution:

Form the semantic tree for the negation of the formula:

- $\neg(A \vee (B \wedge C)) \rightarrow ((A \vee B) \wedge (A \vee C))$
  - $A \vee (B \wedge C)$
    - $\neg((A \vee B) \wedge (A \vee C))$
      - A
        - $\neg(A \vee B)$ — $\neg A$ — $\neg B$
        - $\neg(A \vee C)$ — $\neg A$ — $\neg C$
      - $B \wedge C$ — B — C
        - $\neg(A \vee B)$ — $\neg A$ — $\neg B$
        - $\neg(A \vee C)$ — $\neg A$ — $\neg C$

We see that in each branch there is a propositional symbol and its negation, so each branch closes. Therefore this tree is a semantic proof of $(A \vee (B \wedge C)) \rightarrow ((A \vee B) \wedge (A \vee C))$.

There are other semantic proofs according to the order the rules are applied.

Grading: Max 6 points. Reduce 2 points each time a rule is used incorrectly.

If the semantic proof goes completely wrong, give 1 or 2 points if there are some correct applications of rules.

4. (a) Explain what is meant by soundness of natural deduction.

   (b) Is the propositional formula

$$(p_0 \to (p_1 \vee p_2)) \to (p_0 \to p_1)$$

   derivable by natural deduction?

Solution:

(a) Definitions from the material: "Soundness of deduction means that if we accept some formulas as true and then deduce another formula from them, then also that other formula is true."
"More exactly, soundness of natural deduction means that deductions respect truth in the following sense: If a formula A can be derived from the assumptions $B_1, \ldots, B_n$, and $v(B_1) = \ldots = v(B_n) = 1$ for some valuation v, then also $v(A) = 1$."
One can say also something like
"Natural deduction preserves truth",
"If a propositional formula A is derivable by natural deduction, then it is a tautology", or
"If a propositional formula can be proved by natural deduction from some assumptions, then it is their logical consequence."

(b) We find a valuation that makes the formula false. Therefore soundness guarantees that there is no derivation of
$(p_0 \to (p_1 \vee p_2)) \to (p_0 \to p_1)$. Let us define a valuation v as $v(p_0) = 1$, $v(p_1) = 0$, and $v(p_2) = 0$ (this happens to be the only valuation that makes the formula false). Then we have that $v(p_0 \to (p_0 \vee p_1)) = 1$ and $v(p_0 \to p_1) = 0$ and therefore the implication $(p_0 \to (p_1 \vee p_2)) \to (p_0 \to p_1)$ is not true.

Grading:

(a) Max 3 points. Formal definition in terms of valuations is not necessary. Reduce 1 or 2 points if the explanation is vague or unclear but basically amounts to a right or partially right account. Mixing with completeness, i.e. "a tautology has a natural deduction" is wrong: 0 points. Giving contradictory definition is worth 0 points even if some right elements are in place.

(b) Max 3 points. Some calculation, with a valuation or a truth table, demonstrating that the assumptions can be true while the conclusion false, must be present. Just defining a valuation is not enough, reduce 1 point in that case. The above valuation is the only one that proves the claim. If some calculation error is made leading to a different valuation but the argument still is based on soundness, reduce 1 or 2 points.

5. Using truth table method, determine whether the propositional formula

$$(p_0 \wedge p_1) \rightarrow p_2$$

is a logical consequence of the propositional formula

$$p_0 \wedge (p_1 \rightarrow p_2).$$

Solution:

To show that $(p_0 \wedge p_1) \rightarrow p_2$ is a logical consequence of the propositional formula $p_0 \wedge (p_1 \rightarrow p_2)$, we show that the formula $(p_0 \wedge (p_1 \rightarrow p_2)) \rightarrow ((p_0 \wedge p_1) \rightarrow p_2)$ is a tautology. For this we need to form a truth table (omitted here). From the truth table it can be seen that in every line where $p_0 \wedge (p_1 \rightarrow p_2)$ has value 1, also $(p_0 \wedge p_1) \rightarrow p_2$ has value 1.

Grading: Max 6 points. Reduce 1 or 2 points from simple or more severe errors in the truth table. If the concept of logical consequence is not handled correctly, reduce 3 points.

Notes, tables, or calculators are not allowed in the exam.