University of Helsinki
Introduction to Logic II
Course Examination, May 9, 2022, 2,5 hours
Notes, tables of formulae, and calculators are not allowed in the exam.

1. Using Tarski's truth definition show that the following formula is valid

$$\forall x_0 R_0(x_0, c_0) \to \forall x_0 \exists x_1 R_0(x_0, x_1)$$

Solution:

Solution is based on the definition of validity: A first order formula of vocabulary L is valid if it is satisfied by every assignment in every structure for L.

In this task, the vocabulary $L = \{R_0\}$. For the proof, let us assume that $\mathcal{M}$ is an arbitrary L-structure and s is an assignment.

Now, we apply Tarski's truth definition to the given formula

$\mathcal{M} \models_s \forall x_0 R_0(x_0, c_0) \to \forall x_0 \exists x_1 R_0(x_0, x_1)$

if and only if

$\mathcal{M} \not\models_s \forall x_0 R_0(x_0, c_0)$ or $\mathcal{M} \models_s \forall x_0 \exists x_1 R_0(x_0, x_1)$

Case 1: Assume $\mathcal{M} \not\models_s \forall x_0 R_0(x_0, c_0)$, then the above disjunction is true.

Case 2: Assume $\mathcal{M} \models_s \forall x_0 R_0(x_0, c_0)$.

This means that for all $a \in M$ we have that

$\mathcal{M} \models_{s(a/x_0)} R_0(a, c_0)$.

From this, by substituting the variable x_1 for the constant $c_0^{\mathcal{M}}$, we obtain

For all $a \in M$, $\mathcal{M} \models_{s(a/x_0)(c_0^{\mathcal{M}}/x_1)} R_0(a, x_1)$.

Now we may introduce the existential quantifier, and obtain

For all $a \in M$, $\mathcal{M} \models_{s(a/x_0)} \exists x_1 R_0(a, x_1)$.

Finally, we write the universal quantifier back to the formula

$\mathcal{M} \models \forall x_0 \exists x_1 R_0(x_0, x_1)$, which is what we needed to show in order to have the above disjunction to hold. Therefore, $\mathcal{M}$ and s satisfy the formula $\forall x_0 R_0(x_0, c_0) \to \forall x_0 \exists x_1 R_0(x_0, x_1)$.

Validity is also possible to be show by proving with Tarski's truth definition that the formula $\forall x_0 \exists x_1 R_0(x_0, x_1)$ is a logical consequence of the formula $\forall x_0 R_0(x_0, c_0)$.

Grading: Max 6 points. The solution must be based on Tarski's truth definition but it has not to be as detailed and formal as the above example solution. Reduce 2-3 points from omitting several steps showing how connectives or quantifiers should be handled by the truth definition. For validity, one has to consider an arbitrary L-structure and an assignment, reduce 1 point if that is not mentioned. Reduce 1 or 2 points from unclear presentation. If the solution goes seriously wrong, give 1 or 2 points if there are at least some correct and relevant things related to a solution.

2. Let $\mathcal{M} = (\mathbb{N}, R_0^{\mathcal{M}})$, where $R_0^{\mathcal{M}} = \{(a,b) \in \mathbb{N}^2 \mid a < b\}$.
Show that the set $\{0\}$ can be defined by a formula in $\mathcal{M}$.

Solution:

Suppose the formula A of vocabulary L has only the variable x free. The set P defined by the formula A on a structure $\mathcal{M}$ is the set of elements a such that some (equivalently all) assignment s with $s(x) = a$ satisfies A. We may also write this as $P = \{a \in M \mid \mathcal{M} \models_{s(x/a)} A(x)\}$.

Here the vocabulary is $L = \{R_0\}$. We need to find a formula $A(x)$ of L so that

$$\{0\} = \{a \in M \mid \mathcal{M} \models_{s(x/a)} A(x)\}$$

in the structure of natural numbers with the less than relation, that is $\mathcal{M} = (\mathbb{N}, R_0^{\mathcal{M}})$, where $R_0^{\mathcal{M}} = \{(a,b) \in \mathbb{N}^2 \mid a < b\}$. The solution utilizes the fact that zero is the least element in that ordering.

The most common choices for $A(x)$ are

- $\forall x (R_0(a,x) \lor a = x)$ i.e. a is less than or equal to all other numbers.
- $\neg \exists x R_0(x,a)$ i.e. there is no element less than a.
- Logically equivalent variations of the above two.

Grading: Max 6 points. Presenting a correct formula is enough for a solution. The formula $\forall x R_0(a,x)$ is somewhat close to solution but if one considers it more carefully, one realizes that actually no element satisfies it. Give 3 points.

The formula has to be well formed in the vocabulary $\{R_0\}$, so formulas such as $a < 1$ are not correct, reduce 2-4 points according to how many symbols outside of the vocabulary are used in the defining formula. Reduce 1 or 2 points from unclear presentation. If the solution goes seriously wrong, give 1 or 2 points if there are at least some correct and relevant things related to a solution.

3. Prove by natural deduction $\exists x_0 A \to \exists x_0 B$
   from the formula $\forall x_0(A \to B)$.
   Explicitly check in writing that used quantifier rules apply.

   Solution:

$$\dfrac{\dfrac{[\exists x_0 A]^2 \qquad \dfrac{\dfrac{\dfrac{\forall x_0(A \to B)}{A \to B}\,\forall\text{ E, a} \qquad [A]^1}{B}\,\to\text{ E}}{\exists x_0 B}\,\exists\text{ I, b}}{\exists x_0 B}\,\exists\text{ E, 1, c}}{\exists x_0 A \to \exists x_0 B}\,\to\text{ I, 2}$$

   a: x_0 is free for x_0 in $A \to B$

   b: x_0 is free for x_0 in B

   c: x_0 is not free in $\exists x_0 B$

   Grading: Max 6 points. Reduce two points each time a rule is used incorrectly. Reduce 1 or 2 points if there are uneliminated assumptions or conditions for quantifier rules are left unchecked. If the solution goes seriously wrong, give 1 or 2 points if there are at least some correct and relevant things related to a solution such as correct applications of rules or an adequate beginning of the deduction.

4. Consider the structures $\mathcal{M}$ and $\mathcal{M}'$ for the vocabulary $\{R_0\}$. The universes of both structures $\mathcal{M}$ and $\mathcal{M}'$ consist of the numbers $\{0, 1, 2, 3\}$ and the interpretations of the symbols are

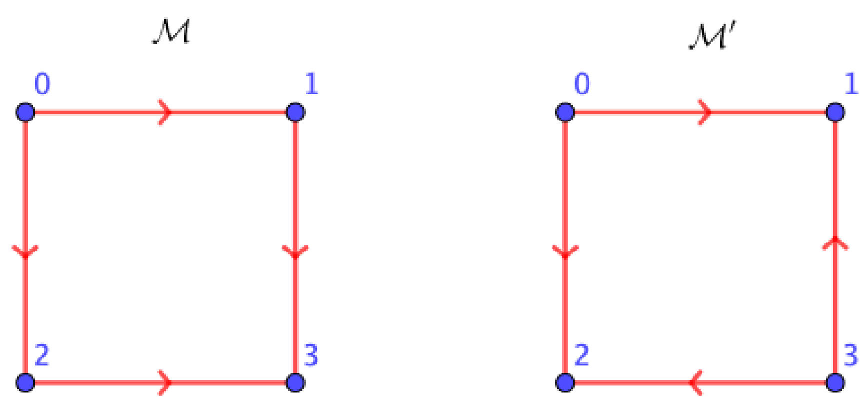
$$R_0^{\mathcal{M}} = \{(0, 1), (0, 2), (1, 3), (2, 3)\}$$

and

$$R_0^{\mathcal{M}'} = \{(0, 1), (0, 2), (3, 1), (3, 2)\}.$$

Are the structures $\mathcal{M}$ and $\mathcal{M}'$ isomorphic?
Complete solution requires explicitly checking the conditions for being isomorphic.

Solution:
Let us first draw pictures of these two graphs:



For these two structures to be isomorphic, it is required that there is a mapping f so that

ISO1 f maps elements of the universe of $\mathcal{M}$ to elements of the universe of $\mathcal{M}'$.

ISO2 Every element of the universe of $\mathcal{M}'$ is the image of exactly one element of the universe of $\mathcal{M}$.

ISO3 $(a, b) \in R_0^{\mathcal{M}}$ if and only if $(f(a), f(b)) \in R_0^{\mathcal{M}'}$ for all $a, b \in M$.

Because the universes of the structures have the same number of elements it is easy to establish conditions ISO1 and ISO2, e.g. by the identity mapping from M to M' with $f(x) = x$ for all $x \in M$. That mapping is injective and surjective, so ISO2 is satisfied.

There are many ways to show that the condition ISO3 cannot hold for these two structures.

For instance, from the picture we see that there are very different "paths" in the two structures (that are in this case directed graphs). In $\mathcal{M}$, starting from 0, it is possible to go via a node and still be able to continue, e.g. with path $(0, 1)(1, 3)$. In $\mathcal{M}'$, that is not the case: from whichever node one starts, it is not possible to continue more than one node. This leads to the observation that it is not possible to map the node 1 in $\mathcal{M}$ to any node in $\mathcal{M}'$ and maintain condition ISO3.

Another way to see that ISO3 cannot hold is to use projections: For example, first projection of $R_0^{\mathcal{M}}$ is $\{0, 1, 2\}$ (the elements for which there is an outbound arrow, whereas first projection of $R_0^{\mathcal{M}'}$ is the set $\{0, 3\}$. Therefore, there could not be an isomorphism between the two structures, since an isomorphism preserves the projections.

Yet another way is to utilize the fact that isomorphisms preserve trurh of formulas. For example, both of the above situations can be expressed by formulas that are true in $\mathcal{M}$ and false in $\mathcal{M}'$.

Grading: Max 6 points. If only the condition ISO3 is checked, reduce 2 points. If ISO1 and ISO2 are correctly checked but the solution argues that ISO3 also holds reduce 3 points. Reduce 1 or 2 points from unclear presentation. If the solution goes seriously wrong, give 1 or 2 points if there are at least some correct and relevant things related to a solution.