University of Helsinki
Master's Programme in Mathematics and Statistics
MAST31201 Mathematical logic, spring 2021
1st mid-term exam
9.3.2021

1. Show directly from the definition of provability (i.e. without the use of any lemmas) that

$$\{\forall v_0(\varphi \to \psi), \forall v_0(\psi \to \theta)\} \vdash \forall v_0(\varphi \to \theta).$$

2. Show that

$$\{\forall v_0(Pv_0 \to Qv_0), \forall v_0 \exists v_1(Pv_0 \to Rv_0v_1)\} \nvdash \forall v_0 \exists v_1(Qv_0 \to Rv_0v_1).$$

3. Consider the $\{P, R\}$-structure $\mathcal{M} = (\{0, 1, 2, 3\}, Sat_{\mathcal{M}})$, where

$$Sat_{\mathcal{M}}(P) = \{0, 1\}$$

and

$$Sat_{\mathcal{M}}(R) = \{(0, 1)(1, 0), (1, 2), (2, 1), (2, 3), (3, 2), (3, 0), (0, 3)\}$$

Is the element 2 definable in $\mathcal{M}$. Justify.

4. (a) Show that if L is a vocabulary, φ and ψ are logically equivalent L-formulas, and t is an L-term such that both $FVF(t, v_0, \varphi)$ and $FVF(t, v_0, \psi)$ hold, then also $\varphi(t/v_0)$ and $\psi(t/v_0)$ are logically equivalent.

   (b) Give an example showing that without the FVF-assumptions the claim does not hold, i.e., substitution into logically equivalent formulas does not necessary produce logically equivalent formulas.