Department of Mathematics and Statistics
Measure and integration
Final exam
10.3.2023

1. Let A and B be measurable subsets of $\mathbb{R}^n$ such that $m(A \cap B) < \infty$. Prove that
$$m(A \cup B) = m(A) + m(B) - m(A \cap B).$$

2. Let $A \subset \mathbb{R}^n$ and let $f\colon A \to \mathbb{R}$ be measurable. Define
$$A_r = \{x \in A\colon f(x) > r\}$$
for $r \in \mathbb{R}$. Suppose that $m_n(A_0) > 0$. Prove that there exists $r > 0$ such that $m_n(A_r) > 0$.

3. Suppose that functions $f\colon A \to \mathbb{R}$ and $g\colon A \to \mathbb{R}$ are measurable. Prove that the sets $\{x \in A\colon f(x) < g(x)\}$ and $\{x \in A\colon f(x) = g(x)\}$ are measurable.

4. Let $f\colon E \to \mathbb{R}$ be a measurable function and E a measurable set such that $m(E) < \infty$. Denote
$$E_i = \{x \in E\colon 0 < f(x) < 1/i\}$$
for $i \in \mathbb{N}$. Prove that
$$\lim_{i \to \infty} m(E_i) = 0.$$

5. Find the limit
$$\lim_{k \to \infty} k \int_1^\infty x^{-3} \sin(x/k) \cos(x/k)\, dx.$$
[Provide an argument for your answer.]