Matematiikan ja tilastotieteen laitos
Mitta ja integraali
Final Exam 5.3.2013

1. Define the Lebesgue outer measure and prove that for every $A \subset \mathbb{R}^n$,

$$m^*(A) = \inf\{m^*(U) : A \subset U \subset \mathbb{R}^n, U \text{ is open}\}.$$

2. Let $E, F \subset \mathbb{R}^n$ be measurable sets such that $E \cap F = \emptyset$, and let A and B be arbitrary sets such that $A \subset E$ and $B \subset F$. Show that

$$m^*(A \cup B) = m^*(A) + m^*(A).$$

3. Define measurable function. Prove that if $f_j : A \to \mathbb{R}, j \in \mathbb{N}$, are measurable functions, then $\limsup_{j\to\infty} f_j$ is measurable.

4. Determine the limit

$$\lim_{k\to\infty} \int_0^1 x^{-1/2} \cos(x^k) e^{-x^2/k} dx.$$

5. Let $f : \mathbb{R} \to \mathbb{R}$ be an integrable function. Show that there exist $x_j \in \mathbb{R}, j \in \mathbb{N}$, such that $\lim_{j\to\infty} x_j = \infty$ and $\lim_{j\to\infty} x_j f(x_j) = 0$.