Model theory
Exam 5.5.2023

Answer 3 of the following 4 questions.

1. Prove Tarki-Vaught i.e. $\mathcal{A} \preceq \mathcal{B}$ if $\mathcal{A} \subseteq \mathcal{B}$ and for all formulas $\phi(v_k, y)$ and $a \in \mathcal{A}^n$, if $\mathcal{B} \models \exists v_k \phi(v_k, a)$ then there is $b \in \mathcal{A}$ such that $\mathcal{B} \models \phi(b, a)$.

2. Let $L = \{E\}$, where E is a binary relation symbol and $T = \{\phi\} \cup \{\phi_n, \psi_n | \ n < \omega\}$ where ϕ says that E is an equivalence relation,

$$\phi_n = \forall v_0 ... \forall v_n \exists v_{n+1} \bigwedge_{i \leq n} \neg E(v_{n+1}, v_i)$$

and

$$\psi_n = \forall v_0 ... \forall v_n \exists v_{n+1} (E(v_{n+1}, v_0) \wedge \bigwedge_{i \leq n} \neg v_{n+1} = v_i).$$

Show that T is complete and has elimination of quantifiers.

3. Suppose that p is an n-type over $\mathcal{A}$. Show that the following are equivalent:
(i) p is consistent i.e. there are $\mathcal{A} \preceq \mathcal{B}$ and $b \in \mathcal{B}^n$ such that b realizes p,
(ii) for all $n < \omega$ and $\phi_i(x, a_i) \in p$, $i \leq n$ (and $x = (v_1, ..., v_n)$),

$$\mathcal{A} \models \exists v_1 ... \exists v_n \bigwedge_{i \leq n} \phi_i(x, a_i).$$

4. Suppose $X = \phi(\mathcal{A}, a) \subseteq \mathcal{A}$, $a \in \mathcal{A}^n$, is strongly minimal. For $A \subseteq X$, let $cl(A) = acl(A \cup a) \cap X$. Show that (X, cl) satisfies Steinitz exchange principle i.e. for all $a, b \in X$ and $A \subseteq X$, if $a \in cl(A \cup \{b\}) - cl(A)$, then $b \in cl(A \cup \{a\})$.