

MODULAR FORMS EXAM 5.5.

Answer all four questions, each 6 points. The reverse side contains results you may assume.

- (1) (a) (1p) Write (no proof required) a fundamental domain for $SL_2(\mathbb{Z})$.
 (b) (1p) Define the valence on $\mathbb{H}$ and at ∞ of a meromorphic modular function f .
 (c) (2p) State the valence formula for $SL_2(\mathbb{Z})$.
 (d) (2p) Sketch the proof of the valence formula for $SL_2(\mathbb{Z})$; especially, explain which part of the contour contributes which term in the formula.

- (2) Let $\Gamma = \Gamma_0(N)$ be the Hecke congruence subgroup of level $N \geq 1$.
 (a) (2p) Define the set of cusps $\mathcal{C}(N)$, and for a cusp s , define its stabiliser Γ_s , its width m_s , and its scaling matrix σ_s .
 (b) (2p) Let f be a modular function of weight k for Γ . Prove that the function $f|_k\sigma_s$ is 1-periodic, that is, for all $z \in \mathbb{H}$

$$(f|_k\sigma_s)(z+1) = (f|_k\sigma_s)(z).$$

- (c) (2p) Define modular forms $M_k(\Gamma)$ and cusp forms $S_k(\Gamma)$.

- (3) Let $f \in S_k(SL_2(\mathbb{Z}))$ with an absolutely convergent Fourier series $f(z) = \sum_{m=1}^{\infty} a_f(m)e(mz)$ for $z \in \mathbb{H}$. Let T_n ($n \geq 1$) denote the Hecke operators for level $N = 1$ and weight k .

- (a) (2p) Prove that

$$T_n f(z) = \sum_{m=1}^{\infty} a_f(m, n) e(mz), \quad \text{where} \quad a_f(m, n) = \sum_{a|\gcd(m, n)} a^{k-1} a_f(mn/a^2).$$

- (b) (2p) Let p and q be distinct prime numbers. Prove that

$$T_p T_q f = T_{pq} f = T_q T_p f.$$

- (c) (2p) Let p be a prime number. Prove that

$$T_p T_p f = T_{p^2} f + p^{k-1} f.$$

- (4) Let $f \in S_k(SL_2(\mathbb{Z}))$ be a Hecke eigenform of level $N = 1$ and even weight $k \geq 2$.

- (a) (1p) Define the L -function $L(f, s)$ and write (no proof) its Euler product.

- (b) (1p) Let $\Lambda^*(f, s) = (2\pi)^{-s} \Gamma(s) L(f, s)$. Prove that for $\text{Re}(s) > (k+1)/2$

$$\Lambda^*(f, s) = \int_0^{\infty} f(iy) y^s \frac{dy}{y}.$$

- (c) (2p) Prove $\Lambda^*(f, s)$ has an analytic continuation with the functional equation

$$\Lambda^*(f, s) = (-1)^{k/2} \Lambda^*(f, k-s).$$

- (d) (2p) Explain (no proof required) what differences in (a),(b),(c) are for $\Gamma_0(N)$, that is, for cusp forms $f \in S_k(\Gamma_0(N))$ of level $N > 1$.

Definition 1 (Weight k slash operator). For $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2^+(\mathbb{R})$, we define

$$(f|_k g)(z) = (\det g)^{k/2} (cz + d)^{-k} f\left(\frac{az + b}{cz + d}\right).$$

Lemma 1 (Argument principle). Let C be a closed simple contour in some open set $U \subseteq \mathbb{C}$ and suppose U contains the interior D of C . Suppose that f is meromorphic in U and that $v_p(f) = 0$ for all $p \in C$. Then as a contour integral integrated counter-clockwise,

$$\frac{1}{2\pi i} \int_C \frac{f'}{f}(z) dz = \sum_{p \in D} v_p(f).$$

Lemma 2 (Argument principle for partial arcs). Let f be meromorphic at a point p and let $C(p, r, \theta)$ be a circular arc of center p , radius r , and angular width θ . Then integrated counter-clockwise,

$$\lim_{r \rightarrow 0} \frac{1}{2\pi i} \int_{C(p, r, \theta)} \frac{f'}{f}(z) dz = \frac{\theta}{2\pi} v_p(f).$$

Lemma 3. For a meromorphic modular function f of weight k for the group $\mathrm{SL}_2(\mathbb{Z})$, for all $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ and all $p \in \mathbb{H}$ the valence satisfies

$$v_{\gamma p}(f) = v_p(f).$$

Lemma 4. The modular group $\mathrm{SL}_2(\mathbb{Z})$ acts transitively on $\mathbb{P}^1(\mathbb{Q}) = \mathbb{Q} \cup \{\infty\}$ by Möbius transformations $r \mapsto \frac{ar+b}{cr+d}$, and the stabiliser of ∞ is $\langle \pm T \rangle$.

Definition 2. Let $N, n \geq 1$ and $\Gamma = \Gamma_0(N)$. Let

$$H_n(N) = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} : a, d \in \mathbb{N}, b \in \mathbb{Z}, ad = n, \gcd(a, N) = 1, 0 \leq b < d \right\}$$

and let $f : \mathbb{H} \rightarrow \mathbb{C}$ and let k be an even integer. We define the Hecke operator $T_n = T_n^{\Gamma, k}$ of level N and weight k by the formula

$$T_n f := n^{k/2-1} \sum_{\rho \in H_n(N)} f|_k \rho.$$

Definition 3. Define the Γ -function by

$$\Gamma(s) = \int_0^\infty x^s e^{-x} \frac{dx}{x}, \quad \operatorname{Re} s > 0.$$

We have $\Gamma(s+1) = s\Gamma(s)$ and for $\operatorname{Re}(s) > -n$ (where n is a positive integer) we define

$$\Gamma(s) = \frac{1}{s(s+1)(s+2)\cdots(s+n-1)} \Gamma(s+n).$$

Lemma 5. We have $\Gamma(s) \neq 0$ for all $s \in \mathbb{C}$.

Lemma 6. Let $f \in S_k(\Gamma_0(N))$ with Fourier coefficients $a_f(n)$. Then for $\sigma > (k+1)/2$

$$\sum_{n=1}^{\infty} \frac{|a_f(n)|}{n^\sigma} < \infty.$$