Partial Differential Equations I (Osittaisdifferentiaaliyhtälöt I)
Exam II (Tentti II)
klo 14:00–16:30, 09.05.2023

Notation

- $B(0,1) = \{x \in \mathbb{R}^n : |x| < 1\}, n \geq 2$. $\partial B(0,1)$ is the boundary of $B(0,1)$ and $\overline{B(0,1)}$ the closure of $B(0,1)$.
- $\Delta u = \sum_{i=1}^{n} u_{x_i x_i}$.

1. Prove that $u = 0$ is the only smooth solution of the following initial-boundary problem

$$\begin{cases} u_t - \Delta u = 0 & \text{in } B(0,1) \times (0,T); \\ u = 0 & \text{on } \Gamma_T = (B(0,1) \times \{t=0\}) \cup (\partial B(0,1) \times [0,T]), \end{cases}$$

where $T > 0$.

2. Let u be a smooth solution of the following initial-boundary problem

$$\begin{cases} u_t - \Delta u = u & \text{in } B(0,1) \times (0,T); \\ u = g & \text{on } \Gamma_T = (B(0,1) \times \{t=0\}) \cup (\partial B(0,1) \times [0,T]), \end{cases}$$

where $T > 0$ and g is a continuous function. Prove that

$$|u(x,t)| \leq e^t \max_{\Gamma_T} |g|$$

for all $(x,t) \in B(0,1) \times (0,T)$.

3. Solve the problem ($u = u(x,t) : \mathbb{R} \times [0,\infty) \to \mathbb{R}$):

$$\begin{cases} u_{tt} - u_{xx} = 0 & \text{in } \mathbb{R} \times (0,\infty); \\ u(x,0) = x; \\ u_t(x,0) = 1, & x \in \mathbb{R}. \end{cases}$$

4. Solve the problem ($u = u(x,t) : \mathbb{R}^3 \times [0,\infty) \to \mathbb{R}$):

$$\begin{cases} u_{tt} - \Delta u = 0 & \text{in } \mathbb{R}^3 \times (0,\infty); \\ u(x,0) = |x|^2; \\ u_t(x,0) = 0, & x \in \mathbb{R}^3. \end{cases}$$