Partial Differential Equations I (Osittaisdifferentiaaliyhtälöt I)
Exam I (Tentti I)
klo 14:00–16:30, 09.03.2023

Notation

- $B(0,1)$ is the unit ball in $\mathbb{R}^n$, $\partial B(0,1)$ its boundary and $\overline{B(0,1)}$ its closure. We denote by α_n the volume of $B(0,1)$.
- $\Delta u = \sum_{i=1}^n u_{x_i x_i}$.

1. Solve the following problem ($u = u(x,t) : \mathbb{R} \times [0,\infty) \to \mathbb{R}$):

$$\begin{cases} 2u_t + u_x = 0; \\ u(x,0) = \sin(2x). \end{cases}$$

2. Solve the following problem ($u = u(x,y) : \mathbb{R}^2 \to \mathbb{R}$):

$$\begin{cases} 2u_x + u_y = 2yu + 2y; \\ u(x,0) = x - 1. \end{cases}$$

3. Let u be a harmonic function in $\mathbb{R}^n, n \geq 2$, such that

$$\int_{\mathbb{R}^n} |u(x)|^2 \, dx < \infty.$$

Prove that u is a constant function.

4. Let $B(0,1)$ be the unit ball in $\mathbb{R}^n, n \geq 3$ and $f \in C(\overline{B(0,1)})$. Suppose that $u \in C^2(\overline{B(0,1)})$ satisfies

$$\begin{cases} -\Delta u \leq f & \text{in } B(0,1); \\ u = 0 & \text{on } \partial B(0,1). \end{cases}$$

Prove that

$$u(0) \leq \frac{1}{n(n-2)\alpha_n} \int_{B(0,1)} \left(\frac{1}{|x|^{n-2}} - 1 \right) f(x) \, dx.$$