Partial Differential Equations I

Spring 2020

Midterm Exam
Thursday March 12.

1. (a) Explain what the weak and strong maximum principles tell about the behaviour of harmonic functions.
   (b) Explain the claim of the Mean Value Principle for harmonic functions.

2. Assume that u is harmonic in the unit disk $D = \{x \in \mathbb{R}^2; |x| < 1\}$ of the plane, continuous in $\overline{D}$ and satisfies the boundary conditions

$$u(e^{i\theta}) = \begin{cases} \sin^2(2\theta), & |\theta| \leq \pi/2 \\ 0, & \pi/2 < |\theta| \leq \pi, \end{cases}$$

where $e^{i\theta} = \cos(\theta) + i\sin(\theta)$, $|\theta| \leq \pi$, are the boundary points of the unit disk.

   (a) Evaluate $u(0)$.
   (b) Prove that at every $x \in D$ it holds that $0 < u(x) < 1$.

3. (a) Let $u \in C^2(\Omega)$, where $\Omega \subset \mathbb{R}^n$ is open. Assume, that $\Delta u(x) > 0$ for all $x \in \Omega$. Can the function u have a strict local maxima in Ω? What about a strict local minima?
   (b) Assume that $u \in C^2(\Omega)$, $\Omega \subset \mathbb{R}^d$ open, solves the Helmholtz equation

$$\Delta u - u = 0.$$

   Show that u can't have a positive local maxima, or a negative local minima.

4. (From the homework) Assume that $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$, with $\Omega \subset \mathbb{R}^d$ a bounded C^1–domain, solves the homogenous Robin–problem

$$\begin{aligned} \Delta u &= 0 \quad \text{in } \Omega, \\ \partial_\nu u + \lambda u &= 0 \quad \text{on } \partial\Omega, \end{aligned}$$

where λ is a complex constant. Show that if $\operatorname{Im}(\lambda) \neq 0$, or if λ is real and $\lambda > 0$, we must have $u = 0$.

For a brief memory aid on Green's formulas see the other side!

Memory Aid - Green's formulas: For $\Omega \subset \mathbb{R}^d$ bounded and $f, g : \Omega \to \mathbb{R}$, $F : \Omega \to \mathbb{R}^d$ regular enough one has

$$\int_\Omega \partial_k f \, dx = \int_{\partial\Omega} \nu_k f \, dS$$

$$\int_\Omega \nabla \cdot F \, dx = \int_{\partial\Omega} \langle \nu, F \rangle \, dS$$

$$\int_\Omega \Delta f \, dx = \int_{\partial\Omega} \partial_\nu f \, dS$$

$$\int_\Omega f \Delta g \, dx = \int_{\partial\Omega} f \partial_\nu g \, dS - \int_\Omega \langle \nabla f, \nabla g \rangle \, dx$$

$$\int_\Omega f \Delta g - g \Delta f \, dx = \int_{\partial\Omega} f \partial_\nu g - g \partial_\nu f \, dS$$