Osittaisdifferentiaaliyhtälöt I
Partial Differential Equations I
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Notation

- $B(0,1) = \{x \in \mathbb{R}^n : |x| < 1\}, n \geq 2$. $\partial B(0,1)$ is the boundary of $B(0,1)$ and $\overline{B(0,1)}$ the closure of $B(0,1)$.
- $\Delta u = \sum_{i=1}^n u_{x_i x_i}$.

1. Let $u \in C^2(\overline{B(0,1)})$ be a positive and harmonic function in $B(0,1) \subset \mathbb{R}^n$. Prove that

$$\frac{1-|x|}{(1+|x|)^{n-1}} u(0) \leq u(x) \leq \frac{1+|x|}{(1-|x|)^{n-1}} u(0),$$

for all $x \in B(0,1)$.

2. Prove that $u = 0$ is the only smooth solution of the following initial-boundary problem

$$\begin{cases} u_t - \Delta u = u & \text{in } B(0,1) \times (0,T); \\ u = 0 & \text{on } \Gamma_T = (B(0,1) \times \{t = 0\}) \cup (\partial B(0,1) \times [0,T]), \end{cases}$$

where $T > 0$.

3. Solve the problem ($u = u(x,t) : \mathbb{R} \times [0,\infty) \to \mathbb{R}$):

$$\begin{cases} u_{tt} - u_{xx} = 0 & \text{in } \mathbb{R} \times (0,\infty); \\ u(x,0) = x; \\ u_t(x,0) = 1. \end{cases}$$

4. Solve the problem ($u = u(x,t) : \mathbb{R}^3 \times [0,\infty) \to \mathbb{R}$):

$$\begin{cases} u_{tt} - \Delta u = t & \text{in } \mathbb{R}^3 \times (0,\infty); \\ u(x,0) = x_1; \\ u_t(x,0) = x_2 x_3, & x = (x_1, x_2, x_3) \in \mathbb{R}^3. \end{cases}$$

Antti