Probability theory I - Exam 25.10.2022

The exam lasts 2 hours. Only pen and paper are allowed on the exam.

PROBLEM 1. Let X be a scalar random variable uniformly distributed on $[0;5]$, and let Y be a random variable defined by

$$Y = \begin{cases} 1, & X \le 2; \\ X, & X > 2. \end{cases}$$

Draw the graph of the probability distribution funciton F_Y, and compute $\mathbb{E}Y$.

PROBLEM 2. Let $Z = (X, Y)$ be a two-dimensional standard Gaussian, i.e., a random variable with values in $\mathbb{R}^2$ such that its coordinates X and Y are independent standard Gaussians. Let R and φ be the random variables, taking values in $[0, \infty)$ and $[0, 2\pi)$ respectively, representing Z in polar coordinates, i.e., such that $X = R\cos\varphi$ and $Y = R\sin\varphi$ almost surely. Prove that R and φ are independent.

PROBLEM 3. Let $X_1, X_2, \dots$ be a sequence of scalar random variables with finite expectations, defined on the same probability space. Consider the following statements:

(1) $X_n \to 0$ almost surely;
(2) $\mathbb{E}X_n \to 0$.

Is it true that (1) implies (2)? Is it true that (2) implies (1)? Justify your answers by giving proofs or counterexamples. How do the answers change if one assumes in addition that almost surely, $|X_n| \le 10$ for all n?

PROBLEM 4. Let $X_1, X_2, \dots$ be i. i. d. scalar random variables such that $\mathbb{E}X_n = 1$. Let $S_n := \sum_{i=1}^n X_i$. Compute the limits $\lim_{n\to\infty} \mathbb{P}(\frac{1}{n}S_n \le 0)$ and $\lim_{n\to\infty} \mathbb{P}(\frac{1}{n}S_n \le 2)$. State some additional condition(s) under which one can compute $\lim_{n\to\infty} \mathbb{P}(\frac{1}{n}S_n \le 1)$, and compute that limit (assuming the conditions you have stated).