Probability theory I - Exam 26.10.2023

The exam lasts two hours, 12:00 – 14:00. Only pen and paper is allowed at the exam.

Problem 1. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, and let $\mathcal{A} \subset \mathcal{F}$ be a π-system, and $B \in \mathcal{F}$ an event which is indepdendent of any event $A \in \mathcal{A}$. Prove that B is independent of any event in $\sigma(\mathcal{A})$.

Problem 2. Consider the probability space $(\Omega; \mathcal{F}, \mathbb{P}) = ((0,1), \mathcal{B}((0,1)), \lambda)$, where λ is the Lebesgue measure on the unit interval $(0,1)$, and a sequence of random variables $X_n = \sqrt{n}\mathbb{I}_{(0,\frac{1}{n})}$ on this space. Does X_n converge to 0 almost surely? In probability? In distribution? In L_1? In L_2? Is this sequence tight? Justify your answers.

Problem 3. Let $X_1, X_2, \ldots$ be i.i.d. scalar random variables such that $\mathbb{E}X_1 = \mu$ and $\text{Var}X_1 = \sigma^2 < \infty$. State and prove the weak law of large numbers for the sequence X_n.

Problem 4. Prove that the characteristic function of any scalar random variable is continuous.