Probability theory I - Exam 25.10.2016

PROBLEM 1. Let $X_1, X_2, \ldots$ be independent, identically distributed random variables such that $\mathbb{E}X_i = 0$ and $0 \neq \mathbb{E}X_i^2 < \infty$, and let $S_n = \sum_{i=1}^n X_i$. Compute $\lim_{n\to\infty} \mathbb{P}(S_n > 0)$.

PROBLEM 2. Let X and Y be two independent scalar random variables, such that X has density $\mathbb{I}_{(0;1)}$ and Y has density $e^{-x}\mathbb{I}_{(0;+\infty)}(x)$. (In other words, X is uniformly distributed on $(0,1)$, and Y has exponential distribution with expectation 1.) Compute $\mathbb{P}(X - Y > 0)$.

PROBLEM 3. Let X and Y be independent standard Gaussians (that is, have density $\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}$). Prove that $X + Y$ and $X - Y$ are independent.

PROBLEM 4. Let $X_1, \ldots, X_{10}$ be independent, identically distributed random variables with density $e^{-x}\mathbb{I}_{(0;+\infty)}(x)$. Prove that

$$\mathbb{P}\left(\sum_{i=1}^{10} X_i \geq 20\right) \leq e^{-10(1-\ln 2)}.$$

PROBLEM 5. Let X be a scalar random variable uniformly distributed on $(0;1)$ (that is, the density of X is given by $\mathbb{I}_{(0,1)}$). Consider the function

$$\psi(t) := \mathbb{E}(\tan(tX^2)).$$

Prove that the function ψ is differentiable on $(-\frac{\pi}{2}; \frac{\pi}{2})$, and compute $\psi'(0)$.

PROBLEM 6. Let $X_1, X_2, \ldots$ be scalar random variables such that for all i, one has $\mathbb{P}(X_i \in \mathbb{Z}) = 1$. Assume that the limit

$$L(k) := \lim_{i\to\infty} \mathbb{P}(X_i = k)$$

exists for all $k \in \mathbb{Z}$, and $\sum_{k\in\mathbb{Z}} L(k) = 1$. Prove that the sequence X_i converges in distribution.

PROBLEM 7. Let F_X and F_Y be two probability distribution functions satisfying $F_X(a) \leq F_Y(a)$ for all $a \in \mathbb{R}$. Prove that there exist random variables X', Y', defined on a common probability space, such that $F_{X'} \equiv F_X$, $F_{Y'} \equiv F_Y$, and $X' \geq Y'$ almost surely.

PROBLEM 8. Let X be a scalar random variable, and let $\varphi_X(t)$ be its characteristic function. Prove that $\mathbb{P}(X \in \mathbb{Z}) = 1$ if and only if $\varphi_X(t)$ is 2π-periodic, that is, $\varphi_X(t) = \varphi_X(t + 2\pi)$ for all $t \in \mathbb{R}$.