Probability theory I - Exam 22.10.2019

The exam lasts 3 hours. Only pen and paper are allowed on the exam. Grading: **6 points** are enough to pass; **14 points** are enough to get 5. Points may be awarded for partial soultions, correct ideas etc.; please write everything down.

PROBLEM 1. **(4 points)** Let $\sigma > 1$, and let ζ and X be two independent scalar random variables such that $\mathbb{P}(\zeta = 1) = \mathbb{P}(\zeta = -1) = \frac{1}{2}$, and X has density $(\sigma - 1)\mathbb{I}_{x\geq 1}x^{-\sigma}$. Compute the expectation $\mathbb{E}(\zeta X)$, depending on σ.

PROBLEM 2. **(4 points)** Let $X_1, X_2, \ldots$ be a sequence of scalar random variables defined on the same probability space. Suppose that there is a constant $C \in \mathbb{R}$ such that $\mathbb{E}|X_i| < C$.

(1) Prove that if $a_n \to 0$, then $a_i X_i \to 0$ in probability;
(2) Prove that if $\sum |a_n| < \infty$, then $a_i X_i \to 0$ almost surely.

PROBLEM 3. **(4 points)** Let $X_1, X_2, \ldots$ be i. i. d. scalar random variables with density $3\mathbb{I}_{x\geq 1}x^{-4}$. Let $S_n := \sum_{i=1}^n X_i$. For each $a \in \mathbb{R}$, compute the limit $\lim_{n\to\infty} \mathbb{P}(\frac{1}{n}S_n \leq a)$. Justify your answer.

PROBLEM 4. **(4 points)** Prove that, for a non-negative random variable X, one has

$$\mathbb{E}X = \int_0^\infty \mathbb{P}(X > t)dt.$$