Probability theory I - Exam 10.01.2018

The exam lasts 4 hours. Only pen and paper are allowed in the exam. Six points are enough to pass, 11 points are enough to get grade 5. You may write your answers in Finnish or English. Points may be awarded also for partial solution, ideas etc.; please do write everything down.

PROBLEM 1. **(2 points)** Let X, Y and Z be three independent scalar random variables uniformly distributed on $(0;1)$ (that is, with density $\mathbb{I}_{(0;1)}$). Compute $\mathbb{E}(\max(X;Y;Z))$.

Olkoon X, Y ja Z riippumattomia satunnaismuuttujia jotka ovat tasajakautuneita välillä $(0;1)$. (toisin sanoen, joiden tiheysfunktio on $\mathbb{I}_{(0;1)}$). Laske $\mathbb{E}(\max(X;Y;Z))$.

PROBLEM 2. Let $X_1, X_2, \ldots$ be independent scalar random variables with density

$$f(x) = \begin{cases} 2|x|^{-5}, & x \notin [-1;1] \\ 0, & x \in [-1;1] \end{cases},$$

and let $S_n = \sum_{i=1}^n X_i$.

(1) **(2 points)** Let $\frac{1}{2} < \alpha < 1$. Prove that

$$\mathbb{P}(|S_n| \geq n^\alpha) \leq \frac{2}{n^{2\alpha-1}}.$$

(2) **(2 points)** Let $0 < \alpha < \frac{1}{2}$. Compute

$$\lim_{n\to\infty} \mathbb{P}(S_n > n^\alpha).$$

(3) **(2 points)** Let $\varphi(\theta) := \mathbb{E}e^{i\theta X_1}$ be the characteristic function of X_1. Compute $\varphi'(0)$ and $\varphi''(0)$.

Olkoon $X_1, X_2, \ldots$ riippumattomia satunnaismuuttujia, joiden tiheysfunktio on

$$f(x) = \begin{cases} 2|x|^{-5}, & x \notin [-1;1] \\ 0, & x \in [-1;1] \end{cases},$$

ja olkoon $S_n = \sum_{i=1}^n X_i$.

(1) Olkoon $\frac{1}{2} < \alpha < 1$. Osoita, että

$$\mathbb{P}(|S_n| \geq n^\alpha) \leq \frac{2}{n^{2\alpha-1}}.$$

(2) Olkoon $0 < \alpha < \frac{1}{2}$. Laske

$$\lim_{n\to\infty} \mathbb{P}(S_n > n^\alpha).$$

(3) Olkoon $\varphi(\theta) := \mathbb{E}e^{i\theta X_1}$ satunnaismuuttujan X_1 karakteristinen funktio. Laske $\varphi'(0)$ ja $\varphi''(0)$.

PROBLEM 3. **(2 points)** Let $X_1, X_2, \ldots$ be scalar random variables, defined on a common probability space, such that $\sum_{n=1}^{\infty} \mathbb{E}|X_n| < \infty$. Prove that $X_n \to 0$ almost surely.

Olkoon $X_1, X_2, \ldots,$ reaaliarvoisia satunnaismuuttujia samalla todennäköisyysavaruudella, siten, että $\sum_{n=1}^{\infty} \mathbb{E}|X_n| < \infty$. Osoita, että $X_n \to 0$ melkein varmasti.

PROBLEM 4. **(2 points)** Give an example of a sequence $X_1, X_2, \ldots$ of i. i. d. symmetric (i. e., X_i has the same distribution as $-X_i$) scalar random variables such that

$$\mathbb{P}\left(\sum_{i=1}^{n} X_i > n\right)$$

does not converge to zero as $n \to \infty$.

Anna esimerkki jonosta $X_1, X_2, \ldots$ riippumattomia ja identisesti jakautuneita symmetrisiä satunnaismuuttujia (eli X_i:lla ja $-X_i$:lla on sama jakauma) sitten, että

$$\mathbb{P}\left(\sum_{i=1}^{n} X_i > n\right)$$

ei suppene nollaan kun $n \to \infty$.