$\mathbb{P}($

$\mathbb{P}(X \geq a) \leq \frac{\mathbb{E}(X)}{a}$

$E(X) \geq E(X \mathbb{1}_{X \geq a}) \geq a\, \mathbb{E}(\mathbb{1}_{X \geq a}) = a\, \mathbb{P}(X \geq a)$

$\Leftrightarrow \mathbb{P}(X \geq a) \leq \frac{\mathbb{E}(X)}{a}$

Probability theory II - Exam 17.12.2019

The exam lasts 3 hours. Only pen and paper are allowed on the exam. Grading: **6 points** are enough to pass; **14 points** are enough to get 5. Points may be awarded for partial soultions, correct ideas etc.; please write everything down.

PROBLEM 1. (4 points) In a certain city, there are N subway stations arranged in a line, so that they are connected as follows:

$$1 \longleftrightarrow 2 \longleftrightarrow \cdots \longleftrightarrow N$$

At day zero, at the station 1 there is an ice-cream stall. Every day, its owner chooses uniformly at random among the following two or three possibilities: either move the stall to a neighboring station, or stay where it is (there are two choices when the stall is at stations 1 or N, and 3 choices otherwise). Let p_t be the probability that the stall is at station 1 again at day t. Compute the limit $\lim_{t\to\infty} p_t$. Justify your answer.

PROBLEM 2. (4 points) Let X and Y be two independent standard Gaussian random variables, and let $Z = X - 2Y$. Compute

$$\mathbb{E}(e^{-X}|Z).$$

PROBLEM 3. Let $\xi_1, \ldots, \xi_n$ be independent random variables such that

$$\mathbb{P}(\xi_i = -1) = 1 - \mathbb{P}(\xi_i = 1) = p,$$

where $0 < p \leq \frac{1}{2}$, and denote

$$S_n := \xi_1 + \cdots + \xi_n.$$

(1) (2 points) Prove that $S_n - (1-2p)n$ is a martingale with respect to the filtration $\mathcal{F}_k = \sigma(\xi_1, \ldots, \xi_k)$.
(2) (2 points) Is this martingale uniformly integrable?
(3) (4 points) Let $\tau = \min\{n : S_n = 1\}$. Compute $\mathbb{E}(\tau)$.

$\mathbb{P}(X_n \geq an) \leq e^{-n\frac{a^2}{2}}$ maybe

S_n

$p = 1 - \mathbb{P}(\xi_i = 1)$

$p + \mathbb{P}(\xi_i = 1) = 1$

$\mathbb{P}(\xi_1$