Real Analysis I

Fall 2019

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Course Exam, 23 October 2019

Solve **exactly** four of the given five questions. Do not submit answers to more than four questions!

Two hours 45 minutes; only standard writing equipment allowed. Each exercise on a different paper + write down your name, student number and the name of the course on each paper. Use margins on both sides of the paper.

1. Let $0 < p < q < \infty$ and let $A \subset \mathbb{R}^n$ be measurable with $0 < |A| < \infty$. Prove that if $f \in L^q(A)$ then
$$\Big(\frac{1}{|A|}\int_A |f|^p\Big)^{1/p} \le \Big(\frac{1}{|A|}\int_A |f|^q\Big)^{1/q}.$$

2. Let $1 < p < \infty$, $f \in L^p(\mathbb{R}^n)$ and $g \in L^1(\mathbb{R}^n)$. Give the proof of the following inequality:
$$\|f * g\|_p \le \|f\|_p \|g\|_1.$$

3. (a) What does it mean that continuous functions are dense in $L^1(\mathbb{R}^n)$?
   (b) Define the centred Hardy–Littlewood maximal function M. What is the boundedness result for Mf when $f \in L^1$?
   (c) If $f \in L^1$, give a proof of the Lebesgue's differentiation theorem:
$$\lim_{r\to 0} \fint_{B(x,r)} |f(y) - f(x)|\,\mathrm{d}y = 0$$
   for almost every $x \in \mathbb{R}^n$.

4. Let $B = B(x_0, R_0) = \{x \in \mathbb{R}^n : |x - x_0| < R_0\}$ be a fixed open ball centred at x_0 and of radius $R_0 > 0$. Let $R > R_0$. Using the theorems of the course show that there is a smooth function $\varphi \in C^\infty$ so that $0 \le \varphi \le 1$, $\varphi(x) = 1$ for $x \in B$, $\operatorname{spt}\varphi \subset B(x_0, R)$ and
$$|\nabla\varphi(x)| \le \frac{C}{R - R_0}.$$

5. Let $f\colon [0,1] \to [0,2]$ be given by $f(x) = x^2(1 + \sin(1/x))$ if $0 < x \le 1$ and $f(0) = 0$. Let $g\colon [0,2] \to \mathbb{R}$ be given by $g(x) = \sqrt{x}$. Prove that the composition $g \circ f$ is not of bounded variation, but that f and g are absolutely continuous.