Series/Sarjat MAT21002, BSMA1005

Course Examination

May 11, 2022, 2,5 hours

No notes, tables of formulae or calculators are allowed in the exam.

Solve each problem. Justify your answers by presenting steps of reasoning or computations as well as justifications for using known rules and results when needed.

Vastaa tehtäviin suomeksi, jos haluat suomenkielisen suoritusmerkinnän.

1. Determine the interval and radius of convergence of the following power series.
   Määritä seuraavan potenssisarjan suppenemissäde ja suppenemisväli.

$$\sum_{k=1}^{\infty} \frac{x^k}{k\sqrt{k}3^k}$$

The radius of convergence can be found by calculating the limit of consecutive coefficients of the series

$$\lim_{k\to\infty}\left|\frac{a_k}{a_{k+1}}\right| = \lim_{k\to\infty}\frac{3(k+1)\sqrt{k+1}}{k\sqrt{k}} = \cdots = 3$$

The center of convergence is 0.
To determine the interval of the convergence, we need to study the end points of the interval (-3, 3).

When $x = -3$, the series is

$$\sum_{k=1}^{\infty} \frac{(-3)^k}{k\sqrt{k}3^k} = \sum_{k=1}^{\infty} (-1)^k \frac{1}{k\sqrt{k}}$$

which is an alternating series whose terms are positive and decreasing. It converges since

$$\lim_{k\to\infty}\frac{1}{k\sqrt{k}} = 0$$

When $x = 3$, the series is

$$\sum_{k=1}^{\infty} \frac{3^k}{k\sqrt{k}3^k} = \sum_{k=1}^{\infty} \frac{1}{k\sqrt{k}} = \sum_{k=1}^{\infty} \frac{1}{k^{3/2}}$$

Which is a p-series and converges, because $p = 3/2 > 1$.

So, at the endpoints the series also convergence, and we may conclude that the interval of convergence is [-3, 3].

Grading: Max 6 points for correct solution. 3 points for correct radius and 3 points for correct interval. Reduce 1 or 2 points for minor calculation errors.

2. Let / olkoon $\mathrm{f}(x) = e^{x^3}$.

   a. Find the Taylor polynomial $T_2(x; 1)$ of f.
      Määritä funktion f Taylorin polynomi $T_2(x; 1)$.
   b. Compute $T_2(x; 1)$.
      Laske $T_2(x; 1)$.
   c. Estimate the error in your approximation at the point $\mathrm{x} = 0$?
      Kuinka suuri on tekemäsi approksimaation virhe pisteessä $\mathrm{x} = 0$?

a. The Taylor polynomial

$$T_2(x; 1) = \sum_{n=0}^{2} \frac{f^{(n)}(1)}{n!}(x-1)^n$$

We need to calculate first and second derivative at point $x = 1$

$$f'(x) = e^{x^3} 3x^2, \quad f'(1) = 3e, \qquad f''(x) = e^{x^3} 9x^4 + e^{x^3} 6x, \qquad f''(1) = 15e$$

Plugging these numbers in to the above sum and simplifying, we obtain

$$T_2(x; 1) = e + 3e(x-1) + \frac{15e}{2}(x-1)^2$$

b. There is a typo. It should have been $T_2(0; 1)$. This item is left out of grading.

c. The error is easy to determine in this case exactly because $f(0) = 1$ and $T_2(0; 1) = \frac{11e}{2}$. So, the error is

$$|T_2(0; 1) - f(0)| = \frac{11e}{2} - 1$$

Grading: For part a, max 3 points. For part b max 3 points. The approximation error may be also estimated using Lagrange's remainder. Reduce 1 or 2 points for minor calculation errors.

3. Do the following series converge?
Suppenevatko seuraavat sarjat?

$$a)\ \sum_{k=2}^{\infty} \frac{1}{\sqrt[3]{k^2-1}}$$

$$b)\ \sum_{k=2}^{\infty} (-1)^k \frac{1}{(log\ k)^k}$$

a) The series diverges. It can be shown for example by comparison test.
b) The series converges. It can be shown for example by alternating series test.

Grading: Max 3 points for each part. If a wrong conclusion is drawn due to a simple calculation error but the choice and justification of the test were correct, reduce 1 point. Reduce 1-2 points from unclear or unprecise presentation.

4. Does the following series converge uniformly on $\mathbb{R}$?
Suppeneeko seuraava sarja tasaisesti joukossa $\mathbb{R}$?

$$\sum_{k=1}^{\infty} \frac{x}{k(1+kx^2)}$$

Let us estimate

$$\left|\frac{x}{k(1+kx^2)}\right| \leq \frac{x^2}{k+k^2x^2} \leq \frac{x^2}{k^2x^2} = \frac{1}{k^2}$$

The series

$$\sum_{k=1}^{\infty} \frac{1}{k^2}$$

converges as a p-series and therefore by the Weierstrass M-test, the original series converges uniformly on $\mathbb{R}$.

There are other ways to find an estimation that does not depend on the variable x.

Grading: Max 6 points for correct solution. Reduce 1-2 points from unclear or unprecise presentation. If the solution starts with setting up a Weierstrass M-test but otherwise goes seriously wrong, give 2-3 points.