SPECTRAL THEORY / EXAM 1 / 08.03.2022

1. Consider the Hilbert space

$$\ell^2 = \{x = (x_j)_{j=1}^{\infty} : \sum_{j=1}^{\infty} |x_j|^2 < \infty,\ x_j \in \mathbb{C}, j = 1, 2, ...\}$$

with the inner product

$$(x|y) = \sum_{j=1}^{\infty} x_j \overline{y_j}, \quad x = (x_j)_{j=1}^{\infty},\ y = (y_j)_{j=1}^{\infty}.$$

Let $\Gamma = (\gamma_j)_{j=1}^{\infty}, \gamma_j \in \mathbb{C}, j = 1, 2, ...$, be a bounded sequence, that is, there is a positive real number C such that $|\gamma_j| \leq C$ for all $j = 1, 2, ...$ Define the multiplication operator $M_\Gamma : \ell^2 \to \ell^2$ as follows

$$M_\Gamma : (x_j)_{j=1}^{\infty} \mapsto (\gamma_j x_j)_{j=1}^{\infty}, \quad \text{for } (x_j)_{j=1}^{\infty} \in \ell^2.$$

i) Show that $M_\Gamma : \ell^2 \to \ell^2$ is a bounded operator; (2 points)

ii) Show that M_Γ is self-adjoint, i.e. $M_\Gamma = M_\Gamma^*$, if and only if γ_j is a real number for each $j = 1, 2, ...$; (4 points)

iii) Show that M_Γ is compact, if and only if the sequence $(\gamma_n)_{n=1}^{\infty}$ converges to 0. (4 points)

2. Consider the operator $B_0 : D(B_0) \to H, B_0 f(t) = i\frac{d}{dt} f(t)$ for $t \in]0, 1[$, where $H = L^2(]0, 1[)$ and $D(B_0) = C_0^\infty(]0, 1[) \subset H$.

i) Show that $\overline{B_0} = B_1$, where $B_1 : D(B_1) \to H, B_1 f(t) = if'(t)$ (weak derivative) for $t \in]0, 1[$ with $D(B_1) = H_0^1(]0, 1[) \subset H$; (4 points)

ii) Show that $B_0^* = B_2$, where $B_2 : D(B_2) \to H, B_2 f(t) = if'(t)$ (weak derivative) for $t \in]0, 1[$ with $D(B_2) = H^1(]0, 1[) \subset H$. (4 points)

3. Let H be a Hilbert space and $U : H \to H$ be a bounded linear operator. Assume that U is a bijection from H onto H and that $U^{-1} = U^*$, where $U^{-1} : H \to H$ is the inverse of U and $U^* : H \to H$ is the adjoint operator of U. Prove that the spectrum of U lies in the unit circle of the complex plane, that is,

$$\sigma(U) \subset \{\lambda \in \mathbb{C} : |\lambda| = 1\}. \quad \text{(6 points)}$$