SPECTRAL THEORY / EXAM 2 / 09.05.2022

1. Let $\{E_\lambda : \lambda \in \mathbb{R}\}$ be a spectral family in a Hilbert space H such that $E_\lambda = 0$ for all $\lambda \leq 0$ and that $E_\lambda = I$ for all $\lambda \geq 2\pi$. Here I is the identity mapping.

i) Show that the mapping $U : H \to H$

$$Ux = \int_0^{2\pi} e^{i\lambda}\, dE_\lambda x$$

is well defined for all $x \in H$; (2 points)

ii) Show that $U \in \mathcal{L}(H)$, that is, U is bounded; (2 points)

iii) Show that the adjoint operator U^* of U is

$$U^* x = \int_0^{2\pi} e^{-i\lambda}\, dE_\lambda x, \quad x \in H; \quad \text{(2 points)}$$

iv) Show that the inverse U^{-1} of U exists and that $U^{-1} = U^*$. (4 points)

2. Let $T \in \mathcal{L}(H)$ be a bounded self-adjoint operator in a Hilbert space H. Suppose that it is positive, that is, $(Tx|x) \geq 0$ for all $x \in H$. Prove that there is a positive bounded self-adjoint operator $B : H \to H$ such that $B^2 = T$. (6 points)

3. Let $U(t) : L^2(\mathbb{R}) \to L^2(\mathbb{R}), t \in \mathbb{R}$, be the translation group in $\mathbb{R}$, defined as follows

$$(U(t)f)(x) = f(x+t), \quad x \in \mathbb{R}$$

for every $f \in L^2(\mathbb{R})$ and every $t \in \mathbb{R}$.

i) Show that $\{U(t) : t \in \mathbb{R}\}$ is a strongly continuous one-parameter unitary group; (2 points)

ii) Show that $U(t) = e^{itA}$ for $t \in \mathbb{R}$, where A is the linear operator such that $D(A) = H^1(\mathbb{R})$ and $Af = -if'$ (weak derivative) for $f \in D(A)$. (6 points)