Stochastic analysis I, Home Exam, Fall Semester 2020

To be returned by 18.12.2020. You can use the course materials and any book or paper you may find relevant, you can also collaborate with fellow students, and ask the teacher for explanations and hints.

1. Consider a two-dimensional pure-jump Lévy process $X(t) = (X_1(t), X_2(t))$, with Lévy measure on $\mathbb{R}^+ \times \mathbb{R}$

$$\begin{aligned}\ell(dx_1, dx_2) &= \alpha \exp\biggl(-\beta x_1 - \frac{x_1 x_2^2}{2}\biggr) x_1^{-1/2} \mathbf{1}(x_1 > 0) dx_1 dx_2 \\ &= \alpha\sqrt{2\pi} \exp\bigl(-\beta x_1\bigr) x_1^{-1}(x_1 > 0) dx_1 \times \exp\bigl(-\frac{x_1 x_2^2}{2}\bigr)\sqrt{\frac{x_1}{2\pi}} dx_2\end{aligned}$$

with $\alpha, \beta > 0$, where you recognize the Lévy measure of the Gamma process for the first component times a conditional zero mean Gaussian distribution for the second component.

   (a) Use the Lévy Khinchine formula to compute the characteristic function of $X(t)$

$$\mathbb{E}\bigl(\exp(i\theta \cdot X(t))\bigr), \quad \theta \in \mathbb{R}^2$$

   (b) Compute the Laplace transform of $X_1(t)$ and the Fourier transform of $X_2(t)$.

   (c) Compute $\mathbb{E}(X_k(t))$, $k = 1, 2$.

   (d) Show that $\mathbf{P}$-almost surely $X(t)$ has finite total variation on finite intervals, equivalently both components $X_1(t)$ and $X_2(t)$ have finite total variation on finite intervals.

   (e) Compute the expectation of the total variation of X_k in an interval $[0, t]$, $k = 1, 2$.

   (f) Compute the expectation of the quadratic covariation matrix $\mathbb{E}\bigl([X_k, X_\ell]_t\bigr)$ $1 \le k, \ell \le 2$.

   (g) Compute joint quadratic moments

$$\mathbb{E}\bigl(X_k(t) X_\ell(s)\bigr) \quad t \ge s \ge 0, \quad 1 \le k, \ell \le 2.$$

2. In this problem we introduce and study in several steps iterated integrals with respect to the Poisson process N and the compensated Poisson process $\bar{N} = (N - \nu)$. The main idea is to use the Palm measure of the Poisson process when we have to compute expectations of iterated integrals and of their squares.

   Finally we establish also an isometry with the iterated Wiener integrals with respect to the Gaussian white noise. Answering the questions marked by $*$ and $**$ is not required for passing the exam. On the other hand it would be good if you could make an additional effort to see the full picture.

   Let $\nu(dt)$ be a σ-finite measure on a Borel space $(\mathbf{T}, \mathcal{B})$, to fix the ideas if you want you can assume that $\mathbf{T} = \mathbb{R}^d$ the euclidean space equipped with its Borel σ-algebra.

(a) For a non-negative and jointly measurable function $f(t_1,\dots,t_n)$ on the n-fold product of measurable spaces $(\mathbf{T}^n,\mathcal{B}^{\otimes n})$, prove that

$$\int_{\mathbf{T}^n} f(t_1,\dots,t_n)\nu(dt_1)\dots\nu(dt_n)=\int_{\mathbf{T}^n}\widetilde{f}(t_1,\dots,t_n)\nu(dt_1)\dots\nu(dt_n)$$

where the normalized sum over the permutations π of $\{1,2,\dots,n\}$

$$\widetilde{f}(t_1,\dots,t_n)=\frac{1}{n!}\sum_{\pi} f(t_{\pi(1)},\dots,t_{\pi(n)})$$

is called the *symmetrization* of f.

(b) Let $D_n=\{t=(t_1,\dots,t_n)\in\mathbf{T}^n:\exists j<k \text{ with } \quad t_j=t_k\}$ be the diagonal set in the product space $\mathbf{T}^n$. From now on we assume that the measure ν is σ-finite and **non-atomic**, $\nu(\{t\})=0\ \forall t\in\mathbf{T}$.
Prove that $D_n\in\mathcal{B}^{\otimes n}$.

(c) Prove that

$$\int_{\mathbf{T}^n} f(t_1,\dots,t_n)\nu(dt_1)\dots\nu(dt_n)=\int_{\mathbf{T}^n\setminus D_n} f(t_1,\dots,t_n)\nu(dt_1)\dots\nu(dt_n)$$

or alternatively that $\nu^{\otimes n}(D_n)=0$ under the product measure $\nu^{\otimes n}=\underbrace{\nu\otimes\nu\otimes\dots\otimes\nu}_{n \text{ times}}$.

Hint Since $\mathbf{T}$ is a Borel space, it admits a measurable bijection with measurable inverse with a measurable subset of the unit interval, you can work with $\mathbf{T}=[0,1]$ and $\mathcal{B}$ is the Borel σ-algebra, which is generated by the dyadic intervals.

(d) Let $N(dt,\omega)$ be a Poisson process driven by a σ-finite non-atomic measure $\nu(dt)$ on the measurable space $(\mathbf{T},\mathcal{B})$, and $f(t_1,\dots,t_n)$ non-negative and jointly measurable function.

(e) Show that the iterated integral with respect to the Poisson process is well-defined:

$$\int_{\mathbf{T}^n} f(t_1,t_2,\dots,t_n)N(dt_1)N(dt_2)\dots N(dt_n)$$

Compute

$$\mathbb{E}\left(\int_{\mathbf{T}^n} f(t_1,t_2,\dots,t_n)N(dt_1)N(dt_2)\dots N(dt_n)\right)$$

Hint: You can start with $n=2$, using the Palm probability of the Poisson process. You can also replace f by its symmetrization $\widetilde{f}$ (why ?). What happens on the diagonals, at points t with $t_j=t_k$ for some $k\neq j$?

(f) Show also that for $\mathbb{R}$-valued $f\in L^1(\mathbf{T}^n,\nu^{\otimes n})$

$$\mathbb{E}\left(\int_{\mathbf{T}^n\setminus D_n} f(t_1,t_2,\dots,t_n)\bar{N}(dt_1)\bar{N}(dt_2)\dots\bar{N}(dt_n)\right)=0$$

where $\bar{N}(B)=N(B)-\nu(B)$ is the compensated Poisson process.

(g) Let $N(dt, \omega)$ be a Poisson process on the unit interval $\mathbf{T} = [0, 1]$ driven by the Lebesgue measure $\nu(dt) = dt$.
For which $\alpha \in \mathbb{R}$

$$\int_0^1 \int_0^1 |t - s|^\alpha N(dt)N(ds) < \infty \qquad \mathbf{P}\text{-almost surely ?}$$

(h) For such α compute also the expectation

$$\mathbb{E}\Big(\int_0^1 \int_0^1 |t - s|^\alpha N(dt)N(ds) \Big)$$

(i) For which $\beta \in \mathbb{R}$

$$\int_{[0,1]^2 \setminus D_2} |t - s|^\beta N(dt)N(ds) < \infty \qquad \mathbf{P}\text{-almost surely ?}$$

(j) For such β, compute the expectation

$$\mathbb{E}\Big(\int_{[0,1]^2 \setminus D_2} |t - s|^\beta N(dt)N(ds) \Big)$$

(k) We want now to define iterated L^2-integrals w.r.t. the compensated Poisson process. The first step is the case with $n = 2$, and use again the Palm measure to compute the squared L^2 norm

$$\mathbb{E}\Big(\Big\{ \int_{\mathbf{T}^2 \setminus D_2} f(s,t) \bar{N}(ds)\bar{N}(dt) \Big\}^2 \Big) =$$
$$= \mathbb{E}\Big(\int_{\mathbf{T}^2 \setminus D_2} \int_{\mathbf{T}^2 \setminus D_2} f(s,t) f(u,v) \bar{N}(ds)\bar{N}(dt)\bar{N}(du)\bar{N}(dv) \Big)$$

when $f(s,t)$ is both in $L^1(\mathbf{T}^2, \mathcal{B}^{\otimes 2}) \cap L^2(\mathbf{T}^2, \mathcal{B}^{\otimes 2})$
Note that the inclusion $(\mathbf{T}^2 \setminus D_2) \times (\mathbf{T}^2 \setminus D_2) \supsetneq \mathbf{T}^4 \setminus D_4$ is strict.

(l) For which $\gamma \in \mathbb{R}$ the iterated L^2 integral below with respect to the compensated Poisson process $\bar{N}(B, \omega) = N(B, \omega) - |B|$ with N driven by the Lebesgie measure is well defined ?

$$\int_{[0,1]^2 \setminus D^2} |t - s|^\gamma \bar{N}(dt)\bar{N}(ds)$$

(m) For such γ, compute the squared L^2-norm

$$\mathbb{E}\Big(\Big\{ \int_{[0,1]^2 \setminus D^2} |t - s|^\gamma \bar{N}(dt)\bar{N}(ds) \Big\}^2 \Big)$$

(n) $*$ When needed you may use the approximation lemma below.
Lemma $\forall f(t_1, \ldots, t_n) \in L^2(\mathbf{T}^n, \nu^{\otimes n})$ there exists a sequence

$$f^{(m)}(t_1, \ldots, t_n) = \sum_{\ell=1}^{K_m} g_1^{(m,\ell)}(t_1) g_2^{(m,\ell)}(t_2) \ldots g_n^{(m,\ell)}(t_n) \quad m \in \mathbb{N}$$

and $f^{(m)} \in L^1(\mathbf{T}^n, \nu^{\otimes n}) \cap L^2(\mathbf{T}^n, \nu^{\otimes n})$ $\forall m \in \mathbb{N}$ such that $f^{(m)} \longrightarrow f$ in $L^2(\mathbf{T}^n, \nu^{\otimes n})$. Also if necessary we can choose $g_j^{(m,\ell)}(t_j)$ to be simple measurable functions taking finitely many values with supports of finite ν-measure.

Show that for $f(t_1, \dots, t_n) \in L^2(\mathbf{T}^n, \nu^{\otimes n})$ the iterated L^2-integral w.r.t. the compensated Poisson process $\bar{N}$

$$\int_{\mathbf{T}^n \setminus D_n} f(t_1, \dots, t_n) \bar{N}(dt_1) \dots \bar{N}(dt_n) =$$
$$= \lim_{m \to \infty} \int_{\mathbf{T}^n \setminus D_n} f^{(m)}(t_1, \dots, t_n) \bar{N}(dt_1) \dots \bar{N}(dt_n)$$

is well defined as a limit in $L^2(\mathbf{P})$, meaning that the $L^2(\mathbf{P})$ limit exists and it does not depend on the approximating sequence.

Hint Continue by induction on n the proof from the case with $n = 2$.

(o) $*$ Compute also the squared $L^2(\mathbf{P})$-norm of the iterated integral

$$\mathbb{E}\bigg(\bigg\{ \int_{\mathbf{T}^n \setminus D_n} f(t_1, t_2, \dots, t_n) \bar{N}(dt_1) \bar{N}(dt_2) \dots \bar{N}(dt_n) \bigg\}^2 \bigg)$$

Hint: consider first the case with $n = 2$, and use again the Palm measure to compute

$$\mathbb{E}\bigg(\int_{\mathbf{T}^2 \setminus D_2} \int_{\mathbf{T}^2 \setminus D_2} f(s,t) f(u,v) \bar{N}(ds) \bar{N}(dt) \bar{N}(du) \bar{N}(dv) \bigg)$$

Note that the inclusion $(\mathbf{T}^2 \setminus D_2) \times (\mathbf{T}^2 \setminus D_2) \supsetneq \mathbf{T}^4 \setminus D_4$ is strict.

Hint Continue by induction on n the proof from the case with $n = 2$.

(p) $**$ We can follow the same path we followed to define the iterated integrals w.r.t. $\bar{N}$, to define the iterated Wiener integral with respect to the Gaussian white noise

$$\int_{\mathbf{T}^n \setminus D_n} f(t_1, \dots, t_n) dW(t_1) \dots dW(t_n)$$

with respect to the Gaussian white noise driven by ν, for $f \in L^2(\mathbf{T}^n, \nu^{\otimes n})$

Show that

$$\mathbb{E}\bigg(\bigg\{ \int_{\mathbf{T}^n \setminus D_n} f(t_1, t_2, \dots, t_n) \bar{N}(dt_1) \bar{N}(dt_2) \dots \bar{N}(dt_n) \bigg\}^2 \bigg)$$
$$= \mathbb{E}\bigg(\bigg\{ \int_{\mathbf{T}^n \setminus D_n} f(t_1, t_2, \dots, t_n) W(dt_1) W(dt_2) \dots \bar{W}(dt_n) \bigg\}^2 \bigg)$$

Hint: use the Wick formula for zero-mean jointly Gaussian random variables G_k

$$\mathbb{E}\big(G_1 G_2 \dots G_{2n}\big) = \sum_{\text{pairings}} \prod_{\{k,\ell\}\text{pairs}} \mathbb{E}\big(G_k G_\ell\big)$$

where for an even number of random variables the sum is over the pairing of $\{1, 2, \dots, 2n\}$ into n disjoint pairs and for each pairing we take the product over the pairs of the pair covariance, and $\mathbb{E}\big(G_1 G_2 \dots G_{2n+1}\big) = 0$ when the number of random variables is odd.