Stochastic analysis I, Fall Semester, Home Exam

1. Show that for a standard Gaussian random variable G, the Gaussian integration by part formula $E_P(f(G)G) = E_P(f'(G))$ is valid also for the step function $f(x) = \mathbf{1}(x > t)$, with derivative $f'(x) = \delta_t(x) = \delta_0(x - t)$ is a *distributional derivative* . is not a function but a generalized function (a distribution in analysis language), the Dirac-delta function at t, with the defining property

$$g(t) = \int_{\mathbb{R}} g(x)\delta_t(x)dx = \int_{\mathbb{R}} g(x)\delta_0(x-t)dx = \int_{\mathbb{R}} g(y+t)\delta_0(y)dy$$

for any continuous test function g with compact support. From the probabilistic point of view the measure $\mu(dx) = \delta_t(x)dx$ is simply the probability measure of a deterministic random variable concentrated in the singleton $\{t\}$.

Hint. In order show that the integration by parts formula is correct also in this case, approximate the indicator $f(x) = \mathbf{1}(x > t)$ by the sequence $f_n(x) = \big((x-t)^+ n\big) \wedge 1$ which satisfies $0 \le f_n(x) \le f(x) \le 1\ \forall x$, and it is piecewise linear with derivative $f_n'(x) = n\mathbf{1}\big(t < x \le t + 1/n\big)$. Apply the Gaussian integration by parts to $f_n(x)$ and use the dominated convergence Theorem to take limits, together with continuity property of the Gaussian density.

2. For $t \in \mathbb{R}$ compute the expectations:

(a) $E_P\big(G\mathbf{1}(G > t)\big)$	(d) $E_P\big(G^2\mathbf{1}(G \le t)\big)$	(g) $E_P\big(G^4\mathbf{1}(G > t)\big)$
(b) $E_P\big(G\mathbf{1}(G \le t)\big)$	(e) $E_P\big(G^3\mathbf{1}(G > t)\big)$	
(c) $E_P\big(G^2\mathbf{1}(G > t)\big)$	(f) $E_P\big(G^3\mathbf{1}(G \le t)\big)$	(h) $E_P\big(G^4\mathbf{1}(G \le t)\big)$

Hint: Justify taking the distributional derivative of the indicator in the integration by parts formula.

3. We recall that a compound Poisson process is a Lévy process with finitely many jumps on finite intervals. Let $X(t)$ be a compound Poisson process with intensity $\lambda > 0$ where the jumps are standard Gaussian random variables.

   (a) Write the Lévy measure of $X(t)$.

   (b) Write the Lévy Khinchine formula for the characteristic function of $X(t)$.

   (c) Show that $X(t)$ is a martingale in its own filtration.

   (d) Show that $M(t) = (X(t)^2 - \lambda t)$ is a martingale in the filtration generated by the $X(t)$ process.

   (e) Can we say that $X(t)$ a Gaussian process ? (justify your answer)

   (f) Show that $X(t)$ is a conditionally Gaussian process conditionally on its jump times.

   (g) Compute the moments $\mathbb{E}(X(t)^k)$, $k \in \mathbb{N}$.

(h) Given $0 = t_0 < t_1 \le t_2 \dots \le t_n$ compute also the joint moment

$$\mathbb{E}\big(X(t_1)X(t_2)\dots X(t_n)\big)$$

4. Show that if $X(t)$ is the compound Poisson process with intensity λ and standard Gaussian jumps given above, the process $X_{a,b}(t) = bX(at)$ is a compound Poisson process with intensity $a\lambda$ and the jumps have Gaussian distribution with zero mean and variance b^2.

5. Now consider the truncated series

$$Y^{(m)}(t) = \sum_{n=1}^{m} b_n X^{(n)}(a_n t)$$

and their limit

$$Y(t) = \sum_{n=1}^{\infty} b_n X^{(n)}(a_n t)$$

where $X^{(n)}(t)$ are independent copies of the compound Poisson process with standard Gaussian jumps above, and $0 < a_n \to +\infty$, and $b_n \to 0$ are deterministic sequences.

(a) Under which conditions on (a_n) and (b_n) the series converges to the process $Y(t)$ **P**-almost surely to a Lévy process with finite variation on finite intervals ?

(b) Write the Lévy-Khinchine formula for the characteristic function of $Y(t)$.

(c) Under which conditions on (a_n) and (b_n) the process $Y(t)$ has square summable jumps and the series converges in $L^2(P)$ to a Lévy process with quadratic variation ?

(d) Write the Lévy Khinchine formula for the characteristic function of $Y(t)$ also in this case, and the Lévy Khinchine formula for the characteristic function the quadratic variation $[Y, Y]_t$.

6. We assume that the arrival times of the busses at the Kumpula campus bus-stop form an homogeneous Poisson process on $\mathbb{R}$ with intensity $\lambda > 0$. Passengers arriving to the bus-stop as an homogeneous Poisson process with intensity $\nu > 0$. What is the distribution of the waiting time to the next bus for a passenger arriving to the bus-stop ?

Let $C(t)$ be the number of passengers waiting for the bus at time t.

A bus can take at most m passengers. Assuming hypothetically that our bus-stop would be the first on the bus-transport line, and all busses arriving to the bus-stop are empty, compute the expectation $\mathbb{E}\big(C(t) \wedge m\big)$ at any fixed time t in the stationary regime (the system is stable for ν large enough, you can also show that when ν is too small the number of passengers waiting the bus will grow to infinity in the long run).

Hint.

$$C(t) = C(0) + N(t) - \int_0^t \big(C(s-) \wedge m\big)\, B(ds)$$

where $N(t)$ is the λ-Poisson process counting the arrival of passengers to the bus-stop and $B(t)$ is the ν-Poisson process counting the arrivals of the busses. In the stationary regime $\mathbb{E}(C(t))$ is constant at all fixed times t.

7. Let τ be a random arrival time of a bus. At stationarity, does the distribution of $C(\tau)$ differs from the distribution of $C(t)$ at a pre-fixed time t ? (explain).

   Let $W(0)$ be random time a passenger arriving at time 0 has wait at the bus-stop before catching the bus. Note that in case more than m passengers are waiting he/she may not necessarily fit in the first bus arriving but he/she may need to wait further until he/she can fit into the bus. Use Little formula (Section 2.4) from the lecture notes to relate the expectation $\mathbb{E}\big(C(t)\big)$ to the the expectation $\mathbb{E}_0\big(W(0)\big)$ with respect to the Palm measure conditioned on a passenger arriving at time 0.