UH Stochastic analysis I, Spring 2018, Home Exam

To be returned before June 4th 2018.

The second part of the course, Stochastic Analysis II, will start on tuesday 13.3. The lecture scheduled for monday 12.3 is moved to wednesday 14.3, at 10.15 in lecture room B120.

We recall the integration by parts formula for cadlag functions with finite variation:

$$X(t)Y(t) - X(0)Y(0) = \int_0^t X(s)Y(ds) + \int_0^t Y(s-)X(ds) = \tag{0.1}$$

$$\int_0^t X(s-)Y(ds) + \int_0^t Y(s)X(ds)$$

$$= \int_0^t X(s-)Y(ds) + \int_0^t Y(s-)X(ds) + [X,Y]_t$$

where $[X,Y]_t = \sum_{s\le t} \Delta X(s)\Delta Y(s)$ is the cross variation.

For cadlag functions $X(t)$ with finite variation on compacts and differentiable functions $f(x)$, we have the change of variable formula

$$f(X_t) - f(X_0) = \int_0^t \frac{\partial f}{\partial x}(X(s))X(ds) + \sum_{s\le t}\Big(f(X(s)) - f(X(s-)) - \frac{\partial f}{\partial x}(X(s-))\Delta X(s)\Big) . \tag{0.2}$$

Recall also that if $Y(t)$ is a $\mathbb{F}$-adapted cadlag process with integrable variation on compact intervals

$$E(\mathrm{Var}_Y(t)) = E\Big(\int_0^t |Y(ds)|\Big) < \infty \forall t,$$

then its dual $\mathbb{F}$-predictable projection Y^p (which is also called the compensator of Y exists and $M(t) := (Y(t) - Y^p(t))$ is a $\mathbb{F}$-martingale.

This follows $X(t)$ is $\mathbb{F}$-predictable and

$$E\Big(\int_0^t |X(s)|\,|Y(ds)|\Big) < \infty \forall t,$$

then by definition of dual predictable projection

$$E\Big(\int_0^t X(s)Y(ds)\Big) = E\Big(\int_0^t X(s)Y^p(ds)\Big)$$

In particular for $X(t,\omega)=\mathbf{1}_F(\omega)\mathbf{1}_{(u,v]}(t)$ for arbitrary $F\in\mathcal{F}_u$ and $0\le u\le v$ which is a simple bounded left-continuous $\mathbb{F}$-adapted process it implies

$$0=E\biggl(\int_0^t X(s)M(ds)\biggr)=E\Bigl(\bigl(M(v)-M(u)\bigr)\mathbf{1}_F\Bigr)$$

which is equivalent to the martingale property $E(M_v|\mathcal{F}_u\bigr)=M_u$.

$$(X\cdot Y)^p=\biggl(\int_0^{\cdot} X(s)Y(ds)\biggr)^p=\int_0^{\cdot} X(s)Y^p(ds)$$

and

$$(X\cdot M)_t=\int_0^t X(s)M(ds)=\int_0^t X(s)Y(ds)-\int_0^t X(s)Y^p(ds).$$

is a $\mathbb{F}$-martingale. Note also that if $X(s)$ is a cadlag $\mathbb{F}$-adapted process, it is $\mathbb{F}$-optional and its left limit $X(s-)=\lim_{r\uparrow s}X(r)$ is $\mathbb{F}$-predictable.

Problems

1. (Discrete time embedded into continuous time). Consider in discrete time a process $(X_n:n\in\mathbb{N})$, and a discrete filtration, $(\mathcal{F}_n:n\in\mathbb{N})$, where X_n in not necessarily $\{\mathcal{F}_n\}$-measurable. Assume that X is integrable or more in general locally integrable in the filtration $(\mathcal{F}_n)_{n\in\mathbb{N}}$.

   We imbed the discrete time processes and filtrations, into continous time processes $X(t)$ and filtration $\mathbb{F}=\{\mathcal{F}_t:t\ge 0)$ which are right-continuous, and piecewise constant between the jump times $n\in\mathbb{N}$:

   $$\mathcal{F}_t=\mathcal{F}_n \text{ and } X(t,\omega)=X_n(\omega)\quad \forall t\in[n,n+1),$$

   Use the definition to show that in continuous time, the $\mathbb{F}$-optional and $\mathbb{F}$-predictable projections of the imbedded process X are respectively

   $${}^oX(t)=E\bigl(X_n\big|\mathcal{F}_n\bigr)\quad t\in[n,n+1) \text{ and } {}^pX(t)={}^oX(t) \text{ if } t\notin\mathbb{N}, \text{ and } {}^pX(n)=E\bigl(X_n\big|\mathcal{F}_{n-1}\bigr)$$

   Show that the dual $\mathbb{F}$-optional projection of X is

   $$X_t^o=\sum_{0\le n\le t}E\bigl(X_n-X_{n-1}\big|\mathcal{F}_n\bigr)=\sum_{0\le s\le t}E\bigl(\Delta X(s)|\mathcal{F}_s\bigr)$$

   and the dual $\mathbb{F}$- predictable projection is

   $$X_t^p=\sum_{0\le n\le t}E\bigl(X_n-X_{n-1}\big|\mathcal{F}_{n-1}\bigr)=\sum_{0\le s\le t}E\bigl(\Delta X(s)|\mathcal{F}_s)_-=\sum_{0\le s\le t}\lim_{r\uparrow s}E\bigl(\Delta X(s)|\mathcal{F}_r\bigr)$$

2. Consider a Bernoulli counting process in the discrete time filtration $\{\mathcal{F}_k : k \in N\}$

$$N(k) = \sum_{i=1}^{k} X_k(\omega)$$

where

$$P\big(X_i = 1 \big| \mathcal{F}_{i-1}\big)(\omega) = 1 - P\big(X_i = 1 \big| \mathcal{F}_{i-1}\big)(\omega) = p \in (0,1)$$

- Imbed now the Bernoulli process N and the filtration in continuous time, and compute the projections ${}^oN, {}^pN, N^o, N^p$.
- For $a > 0$ show that

$$a^{N_n} = 1 + \sum_{j=1}^{n} (a-1) a^{N_{j-1}} \Delta N_j.$$

compute its martingale decomposition to compute its dual predictable projection and use it to compute $\mathbb{E} a^{N_n}$.

3. A stochastic process $X(t)$ is <u>stochastically continuous</u>, if for each t and $\epsilon > 0$:

$$\lim_{h \to 0} P\{|X_{t+h} - X_t| > \epsilon\} = 0.$$

Show that the Bernoulli process (imbedded to continuous time) is not stochastically continuous.

4. Show that any increasing process X, $X_t \in L(P)$, such that the expectation map $t \mapsto EX_t$ is continuous, is stochastically continuous. Show that the Poisson process on $\mathbb{R}_+$ is stochastically continuous.

5. A piecewise constant cadlag process $N(t)$ with $N(0) = 0$ and $\Delta N(t) \in \{0, 1\}$ $\forall t$ is called a *counting process.*

   (a) For the next questions we assume that N is $\mathbb{F}$-adapted and

   $$P(N_t < \infty) = 1 \quad \forall t \geq 0.$$

   Show that an $\mathbb{F}$-adapted counting process is locally bounded. Hint: find a sequence of stopping times $\tau_n(\omega) \uparrow \infty$ such that $N_{\tau_n \wedge t}(\omega) \leq C_n$ $\forall \omega, t$, with constants $C_n < \infty$.

   (b) Assume that N is $\mathbb{F}$-adapted and $P(N_t < \infty) = 1$, and show that the dual predictable projection N^p exists, and

   $$\Delta N^p(t) = \big(N^p(t) - N^p(t-)\big) \in [0, 1].$$

(c) Show also that

$$P\{N_t \geq \epsilon\} \leq EN_t^p, \forall\, \epsilon > 0.$$

(d) Prove that $N(t)$ is stochastically continuous if and only if $t \mapsto N_t^p$ is continuous.

6. Assume that $(Y_k)k \geq 1$, with $Y_0 = 0$, is a sequence of independent Bernoulli random variables with parameter p_k: $P\{Y_k = 1\} = p_k$. Define a counting process N on $[0,1)$ by

$$N_t \doteq \sum_{k=0}^{\lfloor \frac{1}{1-t} - 1 \rfloor} Y_k$$

and put $N_1 = \lim_{t \to 1} N_t$. Show that the process N is non-exploding at the time $t = 1$ if and only if $\sum_k p_k < \infty$. Non-exploding: $P(N_1 < \infty) = 1$. Show also that if N is non-exploding at $t = 1$, then $EN_1 < \infty$.

7. Assume that F is continuous and f is right-continuous with bounded variation. Show that $t \mapsto F(f(t))$ is right-continuous. If $F \in C^1$, i.e. F is differentiable with a continuous derivative, then $F \circ f$ has bounded variation on compacts

8. Let $N(t)$ be a Poisson process with intensity $\lambda > 0$, with cadlag trajectories and a filtration $\mathbb{F} = (\mathcal{F}_t : t \geq 0)$ such that $M(t) = (N(t) - \lambda t)$ is a $\mathbb{F}$-martingale.

   (a) What are the $\mathbb{F}$-optional and $\mathbb{F}$-predictable projections ${}^oN$ and ${}^pN$?

   (b) Show that $N(t)$ has dual $\mathbb{F}$-optional and dual $\mathbb{F}$-predictable projections, find N^o and N^p.

9. We compute the Laplace transform of the λ-Poisson process using martingales.

   For $\theta > 0$ let $f(x) = \exp(-\theta x)$, and use the change of variable formula for cadlag functions $X(t)$ with finite variation on compacts and differentiable $f(x)$

$$f(X_t) - f(X_0) = \int_0^t \frac{\partial f}{\partial x}(X(s))X(ds) + \sum_{s \leq t} \big(f(X(s)) - f(X(s-)) - \frac{\partial f}{\partial x}(X(s-))\Delta X(s)$$

   and a martingale argument to compute the Laplace transform $\theta \mapsto E\big(\exp(-\theta N(t))\big)$, $\theta > 0$ of the λ-Poisson process $N(t)$.

10. **A theorem by Thomas Kurtz** In this exercise we use a martingale argument together with the change of variable formula (0.2) to compute Laplace transforms.

Let $\tau_1, \dots \tau_m$ $\mathbb{F}$-stopping times, and let $N_j(t) = \mathbf{1}(\tau_j \le t)$. Let $\Lambda_j = N_j^p$, the compensator or dual $\mathbb{F}$-predictable projection of N_j (which exists, why ?)

We assume that

$$P(\tau_i = +\infty) = P(\tau_i = \tau_j) = 0 \quad \forall i \neq j$$

and that the compensators $\Lambda_j(t)$ are continuous processes.

We show that the stopped compensators $\Lambda_1(\tau_1), \dots, \Lambda_m(\tau_1)$ are i.i.d. 1-exponential random variables, i.e.

$$P\Big(\Lambda_1(\tau_1) > x_1, \dots, \Lambda_1(\tau_1) > x_n\ \Big) = \exp\bigg(-\sum_{j=1}^{m} x_j\bigg), \quad x_j > 0$$

In order to show it we compute the joint Laplace transform of $\Lambda_{\tau_1}, \dots \Lambda_{\tau_m}$ and show that for $\forall \theta_j > 0$

$$E\bigg(\exp\bigg(-\sum_{j=1}^{m} \theta_1 \Lambda_j(\tau_i)\bigg)\bigg) = \prod_j \frac{1}{(1+\theta_j)} \tag{0.3}$$

which is the product of the Laplace transform of i.i.d. 1-exponential random variables.

(a) Use the change of variable formula to write an integral representation of

$$\zeta_j(\theta_j, t) = (1+\theta_j)^{N_j(t)} \exp\big(-\theta_j \Lambda_j(t)\big)$$

and show that if $\theta_j > 0$, $\zeta_j(t)$ is an uniformly integrable F-martingale.

(b) Show that

$$[\zeta_i(\theta_i), \zeta_j(\theta)]_t = \sum_{s \le t} \Delta\zeta_i(\theta_i, s)\Delta\zeta_j(\theta_j, s) = 0 \quad \forall i \neq j$$

Hint:

$$[N_i, N_j]_t = \sum_{s \le t} \Delta N_i(t) \Delta N_j(t) = 0 \quad \forall i \neq j$$

(c) Use the integration by part formula for product of finite variation processes, to find an integral representation for the product

$$Z(\theta, t) = \prod_{j=1}^{m} \zeta_j(\theta_j, t)$$

and show that $\forall \theta = (\theta_1, \ldots, \theta_m) \in \mathbb{R}_+^m$, $Z(\theta, t)$ is also an uniformly integrable martingale.

(d) Compute $E\big(Z(\theta, \infty)\big)$ to prove (0.3)