Stochastic analysis II, Fall 2019, Home Exam

1. Let $(X_t^{(1)},\dots,X_t^{(n)} : t\ge 0)$ continuous martingales with $X_0^{(j)}=0$ and predictable covariation

$$\langle X^{(j)}, X^{(k)}\rangle_t = c_{jk}\, t, \quad t\ge 0$$

where (c_{jk}) is a positive definite (covariance) matrix.

   (a) Show that $B^{(j)}(t)=c_{jj}^{-1/2}X_t^{(j)}$ are Brownian motions.

   (b) Use inductively Ito formula and Fubini Theorem to compute the joint moment at time t:

$$E_P\big(X_t^{(1)}\dots X_t^{(n)}\big) = \begin{cases} 0 & \text{if } n \text{ is odd} \\ t^{n/2} \sum\limits_{\text{pairings}} \prod\limits_{pairs\{i,j\}} c_{ij} & \text{if } n \text{ is even} \end{cases}$$

   where when n is even, the sum is over all pairings of $1,\dots,n$ into $n/2$ pairs, where the pairs are disjoint and the elements of the pairs are distinct, and for each pairing we then take the product over the pairs of the pairing.

   This is Wick's formula for zero mean jointy Gaussian random variables (in the literature and in the lecture notes you can find a proof based on the moment generating function).

   Hints: Compute the semimartingale decomposition of the product $X_t^{(1)}\dots X_t^{(n)}$, and show that the local martingale part has to be a true martingale with zero expectation.

   Hint: Let's see how Wick formula works in practice, for example

$$E_P\big(X_t^{(1)}X_t^{(2)}X_t^{(3)}X_t^{(4)}\big) = (c_{12}c_{34}+c_{13}c_{24}+c_{14}c_{23})t^2$$

   since we can form disjoint pairs in three possible way, and each pairing contributes with the product of two covariances.

   Another example would be

$$E_P\big((X_t^{(1)})^2X_t^{(2)}X_t^{(3)}\big) = (c_{11}c_{23}+2c_{12}c_{13})t^2$$

   and

$$E_P\big((X_t^{(1)})^2(X_t^{(2)})^2\big) = (c_{11}c_{22}+2c_{12}^2)t^2$$

2. (a) Solve the following Ito SDE

$$a)\quad X_t = x+\int_0^t \sqrt{1-X_s^2}dB_s - \frac{1}{2}\int_0^t X_s ds$$

$$b)\quad X_t = x+\int_0^t \sqrt{1+X_s^2}dB_s + \frac{1}{2}\int_0^t X_s ds$$

$$c)\quad X_t = x+\int_0^t \sqrt{1+X_s^2}dB_s + \int_0^t \big(\sqrt{1+X_s^2}+\frac{1}{2}X_s\big)ds$$

$$b)\quad X_t = x+\int_0^t \exp(-X_s)dB_s + \frac{1}{2}\int_0^t \exp(-2X_s)ds$$

$$c)\quad X_t = x+\frac{1}{3}\int_0^t (X_s)^{1/3}ds + \int_0^t (X_s)^{2/3}dB_s$$

Hint: assume that $X_t = \varphi(B_t)$ and use Ito formula to obtain an equation for φ.
In c) you can assume first that $X_t = \varphi(B_t + a(t))$ and after using Ito formula, choose the function $a(t)$ to simplify the differential equation for φ.

(b) Rewrite the SDE in Stratonovich form.

Remark in general is not always possible to find an explicit solutions of a SDE.

3. Let B_t and W_t two independent Brownian motions under the measure P and let

$$X_t = aW_t + bB_t + ct$$
$$Y_t = \alpha W_t + \beta B_t + \gamma t$$

with deterministic constants $a, b, c, \alpha, \beta, \gamma \in \mathbb{R}$.

Using Girsanov theorem, construct a probability measure Q equivalent to P on finite intervals $[0, t]$ such that both X_t and Y_t are Q-martingales.

Under which conditions on the coefficients $a, b, c, \alpha, \beta, \gamma$ such Q is unique ?

4. We consider a family of linear SDE in Ito sense

$$X_t = x + \int_0^t X_s \theta ds + \int_0^t X_s \sigma dB_s^\theta$$

where (B_t^θ) is Brownian motion under the measure P^θ. We think as $\sigma \neq 0$ fixed, while $\theta \in \mathbb{R}$ is a parameter. Note that

$$B_t^\theta = B_t^0 - \frac{\theta}{\sigma} t$$

where B_t^0 is a Brownian motion under P^0 which corresponds to the value $\theta = 0$.

(a) Compute and the likelihood ratio process

$$Z_t(\theta) = \frac{dP_t^\theta}{dP_t^0}$$

and find a representation as stochastic integral with respect to the integrator (X_t).

(b) Show that $Z_t(\theta)$ is a martingale under P^0.

(c) Compute the logarithmic derivative

$$S_t(\theta) := \frac{d}{d\theta} \log Z_t(\theta)$$

and show that $S_t(\theta)$ is a martingale under P^θ.

(d) Assuming now that the parameter θ is unknown, compute the maximum likelihood estimator $\hat{\theta}_T$ for a given a realization $(X_t(\omega) : t \in [0,T])$. In other words, find the argument $\hat{\theta}$ which maximizes $\log(Z_t(\theta,\omega))$ for the observed realization.

5. (*Brownian bridge and initially enlarged Brownian filtration*) *In this exercise we show that from the point of view of someone knowing the final value $W(T)$ of a Brownian motion $\big(W(t)\big)_{t\in[0,T]}$, the Brownian motion is not a martingale anymore in the bigger filtration and it becomes a Brownian bridge. If you are interested you may look at a couple of books written by Thierry Jeulin and Marc Yor on the theory of enlargement of filtrations.*

   Let $\big(W(t) : t \in [0,T]\big)$ be a Brownian motion.

   (a) *Prove that*

   $$\left(\int_0^T \frac{W(T) - W(s)}{T - s} ds \right) \in L^1(\mathbf{P})$$

   Hint: use Fubini together with the scaling property of Brownian motion.

   (b) *For $0 \le t < T$, compute the conditional density $p_{T,t}(y|x)$ of $W(T)$ conditionally on $W(t)$, such that for every bounded Borel measurable test function f*

   $$M(t,f) = \mathbb{E}\big(f(W(T))\big|\mathcal{F}_t^W\big) = \mathbb{E}\big(f(W(T))\big|W(t)\big) = \int_{\mathbb{R}} f(y) p_{T,t}(y|W(t)) dy$$

   Hint. *Note that since $(W(t), W(T))$ are jointly Gaussian, the conditional density of $W(T)$ given $W(t)$ has to be Gaussian ...*

   (c) *Show that $\forall y \in \mathbb{R}$*

   $$p_{T,t}(y|W(t)) \ for 0 \le t < T$$

   is a non-negative martingale in the open interval $[0,T)$, and compute its representation as a stochastic exponential.

   (d) *Show that however*

   $$p_{T,t}(y|W(t))$$

   is not a martingale in the closed interval $[0,T]$! (the conditional density converges in distribution to a Dirac δ-function.

   (e) *Compute the predictable covariations*

   $$\langle p_{\cdot,T}(y|W(\cdot)), W\rangle_t \ and \ \langle M(t,f), W\rangle$$

   (f) *Show that $0 \le s \le t < T$, $A \in \mathcal{F}_s$ and bounded Borel function $f(y)$,*

   $$\mathbb{E}\Big(\big(W(t) - W(s)\big) f(W(T)) \mathbf{1}_A \Big) \mathbb{E}\Big(f(W(T)) \mathbf{1}_A \int_s^t \frac{W(T) - W(u)}{T - u} du \Big)$$

   Hint. *take the conditional expectation of $f(W_T)$ w.r.t. $\mathcal{F}_t$ and $\mathcal{T}_s$ and rewrite it as integral w.r.t. the conditional density of $W(t)$, use then the properties of the predictable covariation computed above.*

(g) *Show that this is equivalent to*

$$B(t) := W(t) - \int_0^T \frac{W(T) - W(u)}{T - u} du$$

being a martingale w.r.t. $\mathbf{P}$ *in the initially enlarged filtration* $\mathbb{G} = (\mathcal{G}_t : 0 \le t \le T)$ *with* $\mathcal{G}_t = \sigma\big(W(T), W(s) : 0 \le s \le t\big)$. *Note also that*

$$W(t) = B(t) + \int_0^T \frac{W(T) - W(u)}{T - u} du$$

Hint: show that for the enlarged σ*-algebrae we have*

$$\mathcal{G}_s := \mathcal{F}_s \vee \sigma(W_t) = \sigma\big(A \cap \{W(T) \in B\} : \; A \in \mathcal{F}_s, B \in \mathcal{B}(\mathbb{R})\big)$$

(h) *Show that* $B(t)$ *is a Brownian motion independent from the random variable* $W(T)$ *in the initially enlarged filtration* $\mathbb{G}$.

(i) *Let* $B(t)$ *be a Browian motion. For each* $y \in \mathbb{R}$ *the solution of the linear SDE*

$$X(t) = B(t) + \int_0^t \frac{y - X(s)}{T - s} ds, \quad 0 \le t \le T$$

is called Brownian bridge pinned to y *at time* T, *which coincides in law with the distribution of the Brownian motion* $(W(t) : 0 \le t \le T)$ *conditioned on the event* $\{W(T) = y\}$.

(j) *Solve the linear SDE for* $X(t)$,

(k) *show that* $X(t)$ *is Gaussian,*

(l) *compute* $\mathbb{E}(X(t))$ *(depending on* y*)*

(m) *compute Covariance*$(X(t), X(s))$ *for* $0 \le s, t \le T$.

(n) *Show that* $X(T) = y$ *with* $\mathbf{P} = 1$.

(o) *Use Girsanov theorem to compute the Radon-Nikodym derivative between the law of the Brownian bridge* X *and the law of the Brownian motion* B *in any interval* $[0, t]$ *with* $t < T$, *and show that the two laws are equivalent.*

(p) *Show that the law of* B *and the law of* X *are singular in the whole interval* $[0, T]$.

Hint *After solving* (n), *it is not difficult to find an event* $A \in \mathcal{F}_T$ *which has probability* 0 *under the law of* B *and probability* 1 *under the law of* X.