Matemaattisten tieteiden kandiohjelma / MTO

Statistical inference IIa

Course exam 22.12.2023 (duration 2h 30min)

Allowed during the exam: normal writing instruments, a calculator and a handwritten A4 sized cheat sheet.

1. Let $\theta > 0$ be a positive parameter, and let

$$f(y;\theta) = \begin{cases} 3\theta^{-1}y^2\exp(-y^3/\theta), & \text{when } y > 0 \\ 0, & \text{otherwise} \end{cases}$$

Suppose that $Y_1, \ldots, Y_n$ are independent and each of them obeys the above distribution. Let $\mathbf{y} = (y_1, \ldots, y_n)$ a given data set, $y_i > 0$ for every i.

Determine the log likelihood function (log-uskottavuusfunktio) and the maximum likelihood (suurimman uskottavuuden) estimate $\widehat{\theta}$. What is the maximum likelihood estimator of the parameter $\lambda = 2/\theta$?

2. Let $(t_1, t_2, t_3, t_4) = (1, 2, 2, 1)$. Let Y_1, Y_2, Y_3, Y_4 be four independent exponentially distributed random variables and $Y_i \sim \mathrm{Exp}(t_i/\mu)$ where $\mu > 0$. Calculate the Fisher information $i(\mu)$ of the model for parameter μ. Let

$$T = \frac{aY_1 + 2Y_2 + 2Y_3 + Y_4}{2} \quad \text{and} \quad U = \frac{2Y_1 + 2Y_2 + 2Y_3 + 2Y_4}{3}.$$

Show that U is an unbiased estimator of the parameter $g(\mu) = 2\mu$ and determine such a number $a > 0$, for which also T is an unbiased estimator of the parameter $g(\mu)$. Is either one of these unbiased estimators a fully efficient (täystehokas) estimator of the parameter $g(\mu)$? Remember to justify your answer.

3. Let $Y_1, \ldots, Y_n \sim \mathrm{P}(\lambda)$ be independent random variables where $\lambda > 0$ and let $\mathbf{y} = (y_1, \ldots, y_n)$ be a given data, $y_i \in \{0, 1, 2, \ldots\}$ for each i. Calculate the observed information $j(\widehat{\lambda}; \mathbf{y})$ at the ML estimate point and the Fisher information $i(\lambda)$ of the model for parameter λ. What kind of distribution the ML estimator $\widehat{\lambda}(\mathbf{Y})$ obeys asymptotically?

4. Let $Y_1, \ldots, Y_n \sim G(2, 3/\mu)$ be independent, gamma distributed random variables where $\mu > 0$ and let $\mathbf{y} = (y_1, \ldots, y_n)$ be a given data set, $y_i > 0$ for each i. Show that the estimator $\overline{Y} = n^{-1}(Y_1 + \cdots + Y_n)$ is an unbiased, consistent (tarkentuva) and fully efficient (täystehokas) estimator of the parameter $g(\mu) = \frac{2}{3}\mu$.

Distributions:

- Random variable $X \sim \mathrm{G}(\kappa, \lambda)$ has a gamma distribution with parameters $\kappa > 0$, $\lambda > 0$. Its density function is

$$f_X(x; \kappa, \lambda) = \frac{\lambda^\kappa}{\Gamma(\kappa)} x^{\kappa-1} e^{-\lambda x} \mathbf{1}\{x > 0\},$$

expected value $\mathbb{E}X = \kappa/\lambda$ and variance $\operatorname{var} X = \kappa/\lambda^2$. The sum $X_1 + \cdots + X_n \sim \mathrm{G}(\sum \kappa_i, \lambda)$ of gamma distributed independent random variables $X_i \sim \mathrm{G}(\kappa_i, \lambda)$ is also gamma distributed. If $X \sim \mathrm{G}(\kappa, \lambda)$ and $c > 0$ is a constant, then $cX \sim \mathrm{G}(\kappa, \lambda/c)$.

- Random variable $Y \sim \mathrm{Exp}(\lambda)$ has an exponential distribution with parameter $\lambda > 0$. This is a special case of the gamma distribution $\mathrm{Exp}(\lambda) = \mathrm{G}(1, \lambda)$, and its density function is

$$f_Y(y; \lambda) = \lambda e^{-\lambda y} \mathbf{1}\{y > 0\},$$

expected value $\mathbb{E}Y = 1/\lambda$ and variance $\operatorname{var} Y = 1/\lambda^2$. The cumulative distribution function of exponential distribution is $F_Y(y) = (1 - e^{-\lambda y})\mathbf{1}\{y > 0\}$.

- Random variable $Z \sim N(\mu, \sigma^2)$ is normally distributed with parameters μ and $\sigma^2 > 0$. Its density function is thus

$$f_Z(z; \mu, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(z-\mu)^2/2\sigma^2}.$$

Expected value $\mathbb{E}Z = \mu$ and variance $\operatorname{var} Z = \sigma^2$. In addition, $\mathbb{E}(Z - \mu)^4 = 3\sigma^4$ and the odd central moments $\mathbb{E}(Z - \mu)^{2k+1} = 0$.

- A random variable $Z \sim \chi^2_n$ follows the chi squared distribution with degrees of freedom $n > 0$. This is a special case of gamma distribution $\chi^2_n = \mathrm{G}(n/2, 1/2)$ and thus its density function is

$$f_Z(z; n) = \frac{2^{-n/2}}{\Gamma(n/2)} z^{n/2-1} e^{-z/2} \mathbf{1}\{z > 0\},$$

exepected value $\mathbb{E}Z = n$ and variance $\operatorname{var} Z = 2n$.

- Discrete random variable $W \sim \mathrm{P}(\mu)$ obeys Poisson distribution with parameter μ. Its probability mass function is

$$f_W(w; \mu) = \begin{cases} e^{-\mu}\mu^w/w!, & \text{when } w = 0, 1, 2, \dots \\ 0, & \text{otherwise.} \end{cases}$$

Expected value $\mathbb{E}W = \mu$ and variance $\operatorname{var} W = \mu$. The sum of independent Poisson distributed random variables $X_i \sim \mathrm{P}(\mu_i)$ is also Poisson distributed: $X_1 + \cdots + X_n \sim \mathrm{P}(\sum \mu_i)$.