PROBABILITY II A, 2022 – EXAM (27.10)

In this exam, you are allowed a "cheat sheet": namely one hand written A4-sheet (with writing on both sides) as well as a simple calculator (not one capable of symbolic calculation).

Each of the **four** problems below is worth 6 points and if the problem has multiple parts, each part is worth an equal number of points. In the first problem, you do not need to justify your work, but in the other ones, it is essential that you justify your work.

Good luck!

Problem 1: True or false

Which of the following statements are true (you do not need to justify your work)?

(1) Let Ω be a sample space, $\mathbb{P}$ a probability distribution on Ω, $A \subset \Omega$ an event, and $A^{\mathsf{c}} = \Omega \setminus A$ its complementary event. Then $\mathbb{P}(A^{\mathsf{c}}) = 1 - \mathbb{P}(A)$.

(2) Let Ω be a sample space, $\mathbb{P}$ a probability distribution on Ω, $A, B \subset \Omega$ disjoint events, and $\mathbb{P}(B) > 0$. Then

$$\mathbb{P}(A|B) = 0.$$

(3) If two real-valued random variables $X : \Omega_1 \to \mathbb{R}$ and $Y : \Omega_2 \to \mathbb{R}$ have the same distributions $\mathbb{P}_X = \mathbb{P}_Y$, and their expectation values $\mathbb{E}X$ and $\mathbb{E}Y$ exist (and are finite) then the expected values are the same: $\mathbb{E}X = \mathbb{E}Y$.

(4) Let X and Y be independent real-valued random variables, which both have as their distribution the discrete uniform distribution on the set $\{0, 1\}$. Then the distribution of the random variable $Z = X + Y$ is also the discrete uniform distribution on the set $\{0, 1\}$.

(5) The cumulative distribution function of a real-valued random variable is always a continuous function.

(6) Let X be a real-valued random variable whose moment generating function M_X exists on some interval $[-t_0, t_0]$ and $a \in \mathbb{R}$ is a fixed real number. Then the moment generating function of the real-valued random variable $X + a$, namely M_{X+a}, satisfies

$$M_{X+a}(t) = e^{at} M_X(t)$$

for each $t \in [-t_0, t_0]$.

Problem 2: Todennäköisyys ja tunnuslukuja

Let $\lambda > 0$. Let us study the exponential distribution with parameter λ, namely the continuous probability distribution, whose density function is

$$f(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

You can assume that this really defines a probability distribution

Let X be a random variable, whose distribution is this exponential distribution with parameter λ.

(1) Calculate the probability $\mathbb{P}(X \leq 10)$.

(2) Calculate the moment generating function $t \mapsto M_X(t)$ of the random variable X (you do not need to study for which t it exists).

(3) Calculate the expected value $\mathbb{E}X$ (you do not need to justify its existence).

Hint: It might be a good idea to recall how to integrate the exponential function:

$$\int_a^b e^{rx}dx = \frac{1}{r}(e^{rb} - e^{ra})$$

for each $a, b, r \in \mathbb{R}$.

Also, it might be a good idea to notice that you can solve the last part in two ways.

Problem 3: Examples?

Either give an example of the object described below or then justify why such an object does not exist.

(1) A probability distribution on the sample space $\mathbb{N}$, whose probability mass function $(p_k)_{k=0}^{\infty}$, is positive everywhere: $p_k > 0$ for all $k \in \mathbb{N}$.
(2) Two events that are not independent. (Note that if you give an example of such events, you need to also describe the sample space and relevant probability distribution.)
(3) A random variable X, whose variance is zero: $\mathrm{Var}(X) = 0$.

Problem 4: A proof

(1) Let $\lambda > 0$ be a positive real number and $n \geq 1$ a positive integer. Let $X_1, ..., X_n$ be independent exponential random variables with parameter λ, namely the distribution of each random variable X_i is continuous with density function

$$f_{X_i}(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0 \\ 0, & x < 0 \end{cases}.$$

Calculate the cumulative distribution function of the random variable $\max(X_1, ..., X_n)$ (namely calculate $\mathbb{P}(X_1 \leq x, ..., X_n \leq x)$).
(2) Define the random variable $M_n = \max(X_1, ..., X_n) - \frac{1}{\lambda}\log n$. Show that for each $x \in \mathbb{R}$ we have

$$\lim_{n\to\infty} \mathbb{P}(M_n \leq x) = e^{-e^{-\lambda x}}.$$

Hint: in the first part, you might check Problem 2 to remind yourself how to integrate the exponential function. In the second part, you might want to recall that for each $y \in \mathbb{R}$, one has

$$\lim_{n\to\infty} \left(1 - \frac{y}{n}\right)^n = e^{-y}.$$

You may use this fact freely.

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