PROBABILITY II B, 2022 – EXAM (22.12)

In this exam, you are allowed a "cheat sheet": namely one hand written A4-sheet (with writing on both sides) as well as a simple calculator (not one capable of symbolic calculation).

Each of the **four** problems below is worth 6 points and if the problem has multiple parts, each part is worth an equal number of points. In the first problem, you do not need to justify your work, but in the other ones, it is essential that you justify your work.

Good luck!

Problem 1: True or false?

Which of the following claims are true (you do not need to justify your work)?

(1) For a random variable X, for which the expectation $\mathbb{E}e^X$ exists, we have for each $a \in \mathbb{R}$

$$\mathbb{P}(X \geq a) \leq \frac{\mathbb{E}e^X}{e^a}.$$

(2) Let $\Omega = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$ be the unit disk. The function $f : \Omega \to \mathbb{R}$,

$$f(x, y) = \frac{1}{\pi}$$

is the density function of some continuous two-dimensional probability distribution.

(3) Let (X, Y) be a continuous 2-dimensional random vector, whose distribution has density $f_{(X,Y)} : \mathbb{R}^2 \to \mathbb{R}$. The cumulative distribution function of the distribution of (X, Y) can be calculated by

$$F_{(X,Y)}(x, y) = \int_{-\infty}^{x} \left(\int_{-\infty}^{y} f_{(X,Y)}(u, v) dv \right) du.$$

for each $x, y \in \mathbb{R}^2$.

(4) Let (X, Y) be a continuous two-dimensional random vector. Then the distribution of the random variable X is also continuous.

(5) Let (X, Y) be a continuous two-dimensional random vector with density $f_{(X,Y)}$ and let $f_X(x) = \int_{-\infty}^{\infty} f_{(X,Y)}(x, y) dy > 0$ for all $x \in \mathbb{R}$. Also let us assume that the expectation $\mathbb{E}Y$ exists. Then one has

$$\mathbb{E}Y = \int_{-\infty}^{\infty} f_X(x) \left(\int_{-\infty}^{\infty} y f_{Y|X}(y|x) dy \right) dx.$$

(6) In case $X = (X_1, ..., X_n) \sim \mathcal{N}(\mu, \Sigma)$ for some $\mu \in \mathbb{R}^n$ and $\Sigma \in \mathbb{R}^{n \times n}$ (satisfying $\Sigma^\mathsf{T} = \Sigma$ and $x^\mathsf{T} \Sigma x \geq 0$ for all $x \in \mathbb{R}^n$), then

$$\mathrm{Cov}(X_i, X_j) = \Sigma_{i,j}.$$

Problem 2: A calculation

Let $f : [0, 1] \times [0, 1] \to \mathbb{R}$,

$$f(x, y) = 2xy + \frac{3}{2} y^2.$$

(1) Show that f is the density function of some continuous two-dimensional probability distribution.

(2) Let (X, Y) be a random vector whose distribution has density f and let $x \in [0, 1]$. Calculate the marginal density of X, f_X, and show that the conditional distribution of Y given $X = x$ has density

$$f_{Y|X}(y|x) = \frac{2xy + \frac{3}{2}y^2}{x + \frac{1}{2}}.$$

(3) Show that for each $x \in [0, 1]$,

$$\mathbb{E}(Y|X = x) = \frac{\frac{2x}{3} + \frac{3}{8}}{x + \frac{1}{2}}.$$

Problem 3: Examples?

Either give an example of each object described below (and explain why it is an example) or explain why such an object does not exist.

(1) A two-dimensional random vector $(X, Y) \sim \mathcal{N}(\mu, \Sigma)$, for which the random variable X is not normally distributed (here we understand a constant random variable as a normally distributed random variable with variance 0).

(2) A normally distributed two-dimensional random vector (X, Y), whose components X and Y are not independent random variables.

(3) A two-dimensional continuous random vector (X, Y), for which $f_{Y|X}(y|x) = f_Y(y)$ for all $x, y \in \mathbb{R}$. Here $f_{Y|X}(y|x)$ is the density of the conditional distribution of Y given $X = x$ and f_Y is the marginal density of Y.

Problem 4: A proof

Let $(M_n)_{n=0}^{\infty}$ be a sequence of random variables for which $(M_0, ..., M_n)$ is a continuous random vector for each $n \geq 0$ and

$$\mathbb{E}(M_{n+1}|(M_0, M_1, ..., M_n)) = M_n$$

for each $n \geq 0$.

(1) Show that for such a sequence one has

$$\mathbb{E}M_n = \mathbb{E}M_0$$

for each $n \geq 0$.

(2) Show that for such a sequence one has

$$\mathbb{E}M_n^2 \leq \mathbb{E}M_{n+1}^2$$

for each $n \geq 0$.

Hint: you might want to use the "tower property" of conditional expectation, namely to write e.g. in the first part that $\mathbb{E}(M_n) = \mathbb{E}(\mathbb{E}(M_n|(M_0, ..., M_{n-1})))$. In the second part, you might want to use a similar trick and write $M_{n+1} = M_n + (M_{n+1} - M_n)$ as well as try to justify why

$$\mathbb{E}((M_{n+1} - M_n)M_n|(M_0, ..., M_n)) = 0.$$

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