Topology II
Exam October 15, 2024
Exam time 14.00-17.00

Problems

p1. Let $(X, \mathcal{T})$ be a topological space and let $\mathcal{B}$ be a basis of $\mathcal{T}$. Let also $A \subset X$ be a subset and $x \in X$. Show that $x \in \overline{A}$ if and only if for every $B \in \mathcal{B}$ satisfying $x \in B$ holds $B \cap A \neq \emptyset$.

p2. Let $(X, \mathcal{T})$ and $(Y, \mathcal{T}')$ be topological spaces, let $f : X \to Y$ be a continuous map from $(X, \mathcal{T})$ to $(Y, \mathcal{T}')$, and let $\mathcal{T}'_f$ be the topology induced by f from $\mathcal{T}'$. Let also $g : X \to Y$ be a continuous map from $(X, \mathcal{T}'_f)$ to $(Y, \mathcal{T}')$. Show that g is continuous from $(X, \mathcal{T})$ to $(Y, \mathcal{T}')$.

p3. Let I be a set and, for each $i \in I$, let $(X_i, \mathcal{T}_i)$ be a topological space and $E_i \subset X_i$ a dense subset in $(X_i, \mathcal{T}_i)$. Show that $E = \{(x_i)_{i \in I} : x_i \in E_i \text{ for each } i \in I\}$ is a dense subset of the product space $\prod_{i \in I} X_i$.

p4. Show that there exists a continuous function $f : [0, 1] \to \mathbb{R}$ which is not monotone on any subinterval $[a, b]$ of $[0, 1]$ for $0 \leq a < b \leq 1$. (*Hints:* You may want to follow the following strategy[1]: For rational numbers $0 \leq p < q \leq 1$, denote $G_{p,q} = \{f \in C[0, 1] : f \text{ is not monotone on } [p, q]\}$. (a) Show that $G_{p,q}$ is open in $C[0, 1]$ (2p). (b) Show that $G_{p,q}$ is dense in $C[0, 1]$ (2p). (c) Use Baire's theorem (2p).)

$\sin\left(\frac{x-a}{b-a} 2\pi\right) c$

$\mathrm{Sin}\left((x-a) \frac{2\pi}{}\right.$

$f(x) = \frac{x-a}{2\pi}$

$f(x) = (x-a)(x-b)2\pi$

$\frac{\frac{a+b}{2} - a}{b-a} = \frac{\frac{b-a}{2}}{b-a} = \frac{1}{2}$

[1] By Väisälä, we know that the space $C[0, 1]$ of continuous functions $f : [0, 1] \to \mathbb{R}$ with the supremum norm $\|\cdot\|$ is a complete norm space and that the related metric space $(C[0, 1], d)$ is a complete metric space.