Topology II
Exam October 17, 2024
Exam time 14.00-17.00

Problems

p1. Let $(X, \mathcal{T})$ be a topological space and let $\mathcal{B}$ be a basis of $\mathcal{T}$, and let $(Y, \mathcal{T}')$ be a topological space and let $\mathcal{B}'$ be a basis of $\mathcal{T}'$. Let also $f\colon X \to Y$ be a function. Show that f is an open map, if and only if for each $x \in X$ and $B \in \mathcal{B}$ containing x there exists $B' \in \mathcal{B}'$ for which $f(x) \in B' \subset fB$.

p2. Let $f\colon (X, \mathcal{T}) \to (Y, \mathcal{T}')$ be continuous map and let $\mathcal{T}^f$ be the topology coinduced by f from $\mathcal{T}$. Let $g\colon Y \to Z$ be continuous mapping from $(Y, \mathcal{T}')$ to $(Z, \mathcal{T}'')$. Show that $g\colon Y \to Z$ is continuous from $(Y, \mathcal{T}^f)$ to $(Z, \mathcal{T}'')$.

p3. Let I be a set and, for each $i \in I$, let $(X_i, \mathcal{T}_i)$ be a topological space and $E_i \subset X_i$ be a subset having empty interior in $(X_i, \mathcal{T}_i)$, that is, $\operatorname{int}_{X_i} E_i = \emptyset$. Show that $E = \{(x_i)_{i\in I} : x_i \in E_i \text{ for each } i \in I\}$ has empty interior in the product space $X = \prod_{i\in I} X_i$, that is, $\operatorname{int}_X E = \emptyset$.

p4. Show that there is no function $f\colon \mathbb{R} \to \mathbb{R}$ which has a strict local minimum[1] at each point of $\mathbb{R}$. Note that f is not assumed to be continuous. (*Hints:* You may want to follow the following strategy: For each $n \in \mathbb{N}$, let $E_n = \{x \in \mathbb{R} \setminus \mathbb{Q} : f(y) > f(x) \text{ for }]x - 1/n, x + 1/n[\}$. (a) Show that $\mathbb{R} = \bigcup_{n\in\mathbb{N}} \overline{E_n} \cup \bigcup_{q\in\mathbb{Q}} \{q\}$ (2p). (b) Show that there exists $n \in \mathbb{N}$ for which $\operatorname{int}\overline{E_n} \neq \emptyset$ (2p). (c) Show that there exists $q \in \mathbb{Q}$ which is not a local minimum (2p).)

[1] A real valued function $f\colon X \to \mathbb{R}$ on a topological space X has a *strict local minimum at* $x \in X$ if there exists a neighborhood U of x for which $f(y) > f(x)$ for $y \in U \setminus \{x\}$.